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AN UNSOLVABLE COUSIN PROBLEM

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ABSTRACT. We construct a Laurent series $\sum_{n=-\infty}^{\infty} A_n z^n$, convergent in an annulus $\{\rho < |z| < 1/\rho\}$ for some $\rho < 1$, which satisfies an algebraic differential equation (ADE) there, but such that neither of the series $\sum_{n=0}^{\infty} A_n z^n$ and $\sum_{n=-\infty}^{-1} A_n z^n$ satisfies any ADE.

In [**RUB**], we constructed a Fourier series $u(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}$ that satisfies an ADE, but such that the truncated series $\sum_{n=0}^{\infty} A_n e^{in\theta}$ satisfies no ADE on any interval. The construction used, as its main tool, the Riesz-Herglotz factorization of the function $e^{-10z^{1/3}} \sin \pi z^{1/3}/\pi z^{1/3}$, where it was shown that the Blaschke product associated with this (obviously differentially algebraic) bounded analytic function in the right half-plane is *not* differentially algebraic (DA), where we say that a function is DA if it solves some nontrivial ADE, $P(x, y, y', \ldots, y^{(n)}) = 0$, where P is a complex polynomial in its $n+2$ variables. Several applications of these results were obtained, notably a series $\sum a_n \cos nx$ that is DA, but such that $\sum a_n \sin nx$ (same a_n) is not DA on any interval. Also, it was shown that the solution of the Dirichlet problem for Laplace's equation in the unit disc need not be DA even though the boundary values are assumed to be DA. In that paper, the relevance of these results to analog computers was elaborated upon.

In this note, we obtain all the above results, with the exception of the one about there existing a DA bounded analytic function whose Blaschke product factor is not DA, by entirely different means, namely by consideration of the entire function

$$\varphi(z) = \prod_{n=0}^{\infty} \left(1 - \frac{\cosh z}{\cosh n}\right).$$

The properties of this function are in the literature (see [**VAL** and **WIT**]), but at the cost of only a little space, we can develop them here.

THEOREM. *Given ρ with $0 < \rho < 1$, there exists a Laurent series $L(z) = \sum_{n=-\infty}^{\infty} A_n z^n$ which converges for $\rho < |z| < 1/\rho$ and satisfies there an algebraic differential equation, but neither of the series $\sum_{n=0}^{\infty} A_n z^n$ and $\sum_{n=-\infty}^{-1} A_n z^n$ satisfies any algebraic differential equation.*

This may be regarded as saying that a certain additive Cousin problem cannot be solved by a general-purpose analog computer, since (see [**LIR**]), the outputs of such computers may be identified with the functions that are differentially algebraic, at

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least in the analytic case. Here, the Cousin data is the function $L(z)$ in the annulus, and the problem is to represent it as the difference (equivalently, the sum) of two DA functions, the first being analytic inside the outer circle, and the second being analytic outside the inner circle.

PROOF. We will implicitly use throughout this paper the fact (see [**OST**] or [**BOR**]) that the sum, product, composition, compositional inverse, of DA functions is again DA. First of all, following a suggestion of Steven Bank, we show that φ is DA. For φ is an entire function of order 2 that vanishes on the rectangular lattice points $\{n + 2k\pi i\colon n, k \in \mathbf{Z}\}$, which is exactly the zero-set of Weierstrass's sigma-function $\sigma(z)$, another entire function of order 2.

Hence $\varphi(z)$ differs from $\sigma(z)$ just by a factor $\exp P$, where P is a polynomial. But (see [**SAZ**]), $\sigma(z)$ is DA, and consequently $\varphi(z)$ is DA. Let

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z + 1/z}{2^n + 2^{-n}}\right).$$

By the above considerations, $f \in$ DA. Let

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{2^n}\right), \qquad h(z) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{z2^n}\right).$$

Straightforward computation (see [**VAL**, p. 226]) shows that

$$f(z) = Cg(z)h(z), \quad \text{where } C = \prod_{n=1}^{\infty} \left(1 + \frac{1}{2^{2n}}\right)^{-1}.$$

(Just write

$$\begin{aligned}\left(1 - \frac{z}{2^n}\right)\left(1 - \frac{1}{z2^n}\right) &= 1 + \frac{1}{2^{2n}} - \frac{1}{2^n}\left(z + \frac{1}{z}\right) \\ &= \left(1 + \frac{1}{2^{2n}}\right)\left[1 - \frac{z + 1/z}{2^n + 2^{-n}}\right].\end{aligned}$$)

However, $g(z) \notin$ DA. There are numerous ways to see this (see [**WIT**] for example). Here is one way. Note that $g(z)$ satisfies the (Poincaré) functional equation $g(2z) = (1 - z)g(z)$. Writing $g(z) = \sum a_n z^n$, we have

$$2^n a_n = a_n - a_{n-1}, \quad \text{or} \quad a_n = a_{n-1}/(1 - 2^n).$$

Note that the a_n are rational numbers. Roughly, then, $a_n \sim 2^{-n(n-1)/2}$, which by [**POP**, p. 79] are too small to be the coefficients of a DA function. To conclude the proof of the theorem, we just take logarithms after first removing a finite number of zeros from g and h, that is let $\tilde{g}(z) = \prod_{n=N}^{\infty}(1 - z/2^n)$, $\tilde{h}(z) = \prod_{n=N}^{\infty}(1 - 1/z2^n)$, and let $\tilde{f}(z) = \tilde{g}(z)\tilde{h}(z)$. Then $\tilde{f} \in$ DA while $\tilde{g} \notin$ DA, and consequently $\tilde{h} \notin$ DA. Taking N so large that $\tilde{g}$ and $\tilde{h}$ become zero-free in $\{\rho < |z| < 1/\rho\}$, we let

$$\tilde{F}(z) = \log \tilde{f}(z), \qquad \tilde{G}(z) = \log \tilde{g}(z), \qquad \tilde{H}(z) = \log \tilde{h}(z).$$

Then $\tilde{F}$ is holomorphic and DA in $\{\rho < |z| < 1/\rho\}$, while $\tilde{G}$ is holomorphic in $\{|z| < 1/\rho\}$ and $\tilde{H}$ is holomorphic in $\{\rho < |z|\}$, and $\tilde{G} \notin$ DA, $\tilde{H} \notin$ DA. Writing

$$\tilde{F}(z) = \sum_{n=-\infty}^{\infty} A_n z^n,$$

we have proved our result, since

$$\tilde{F}(z) = \tilde{G}(z) + \tilde{H}(z).$$

ADDED IN PROOF. From the theory of elliptic functions, the theta-series

$$S(x) = \sum_{n=-\infty}^{\infty} q^{n^2} x^n$$

satisfies an algebraic differential equation in x of the third order. However, as Pólya observed, (see [**POL I, II**]) the "half" theta-series

$$\varphi(x) = \sum_{n=0}^{\infty} q^{n^2} x^n,$$

for q rational and $0 < q < 1$ satisfies no algebraic differential equation in x. Thus, the main result of our paper follows directly from the classical literature.

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