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A NONLINEAR ERGODIC THEOREM FOR A REVERSIBLE SEMIGROUP OF LIPSCHITZIAN MAPPINGS IN A HILBERT SPACE

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ABSTRACT. Let C be a nonempty closed convex subset of a Hilbert space, S a right reversible semitopological semigroup, $\mathscr{S} = \{T_t : t \in S\}$ a continuous representation of S as Lipschitzian mappings on a closed convex subset C into C, and $F(\mathscr{S})$ the set of common fixed points of mappings T_t, $t \in S$. Then we deal with the existence of a nonexpansive retraction P of C onto $F(\mathscr{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and Px is contained in the closure of the convex hull of $\{T_tx : t \in S\}$ for each $x \in C$.

1. Introduction. Let C be a closed convex subset of a real Hilbert space H and T be a mapping of C into itself. T is said to be a Lipschitzian mapping if for each $n \geq 1$ there exists a positive real number k_n such that

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

for all $x, y \in C$. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \geq 1$ and asymptotically nonexpansive if $\lim_n k_n = 1$, respectively. We denote by $F(T)$ the set of fixed points of T. The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [**1**]: Let C be a closed convex subset of a Hilbert space and T be a nonexpansive mapping of C into itself. If the set $F(T)$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{\text{co}}A$ is the closure of the convex hull of A. In [**11** and **12**], Takahashi proved the existence of such a retraction—"ergodic retraction"—for an amenable and a right reversible semigroup of nonexpansive mappings in a Hilbert space. On the other hand, Hirano and Takahashi [**4**] proved the Cesàro means for asymptotically nonexpansive mappings converge weakly to a fixed point.

In this paper, we deal with the existence of "ergodic retraction" for a right reversible semigroup of Lipschitzian mappings; that is, we prove a nonlinear ergodic theorem for such a semigroup in a Hilbert space. This theorem is a generalization

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of the result [**4**, **12**]. Furthermore, we prove a weak convergence theorem which is similar to that of [**7**, **9**].

2. Nonlinear ergodic theorem. Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. Let C be a nonempty closed convex subset of a Hilbert space H. Then a family $\mathscr{S} = \{T_t : t \in S\}$ of mappings from C into itself is said to be a Lipschitzian semigroup on C if $\mathscr{S}$ satisfies the following:

(1) $T_{ts}(x) = T_t T_s(x)$ for $t, s \in S$ and $x \in C$;

(2) the mapping $(s, x) \to T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology;

(3) for each $s \in S$, there exists $k_s > 0$ such that $||T_s(x) - T_s(y)|| \leq k_s ||x - y||$ for $x, y \in C$.

A semitopological semigroup S is right reversible if any two closed left ideals of S have nonvoid intersection. In this case, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $a \leq b$ if and only if $\{a\} \cup \overline{Sa} \supseteq \{b\} \cup \overline{Sb}$. Let $F(\mathscr{S})$ denote the set $\{x \in C : T_s x = x \text{ for all } s \in S\}$ of common fixed points of mappings T_s, $s \in S$ in C (fixed point theorems for Lipschitzian semigroups were proved in [**6**, **3**]). Then we have the following:

THEOREM 1. *Let C be a closed convex subset of a real Hilbert space H and let S be a right reversible semitopological semigroup. Let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C with* $\limsup_s k_s \leq 1$. *Then $F(\mathscr{S})$ is a closed convex subset of C.*

PROOF. Closedness of $F(\mathscr{S})$ is obvious. To show convexity it is sufficient to prove that $z = (x + y)/2 \in F(\mathscr{S})$ for all $x, y \in F(\mathscr{S})$. We have

$$||T_t z - x||^2 = ||T_t z - T_t x||^2 \leq k_t^2 ||z - x||^2 = \tfrac{1}{4} k_t^2 ||x - y||^2$$

and

$$||T_t z - y||^2 = ||T_t z - T_t y||^2 \leq k_t^2 ||z - y||^2 = \tfrac{1}{4} k_t^2 ||x - y||^2.$$

Thus

$$||T_t z - z||^2 = \tfrac{1}{2} ||T_t z - x||^2 + \tfrac{1}{2} ||T_t z - y||^2 - \tfrac{1}{4} ||x - y||^2 \leq \tfrac{1}{4}(k_t^2 - 1) ||x - y||^2$$

and hence $\lim_t ||T_t z - z|| = 0$. Therefore we obtain

$$z = \lim_t T_t z = \lim_t T_{st} z = \lim_t T_s T_t z = T_s \lim_t T_t z = T_s z$$

for all $s \in S$.

Let $\{x_\alpha : \alpha \in A\}$ be a bounded net of a Hilbert space H and let C be a closed convex subset of H. Then we define

$$r(x) = \lim_\beta \sup_{\beta \leq \alpha} ||x_\alpha - x|| \quad \text{and} \quad r = \inf\{r(x) : x \in C\}.$$

It is well known that there exists a unique element $a \in C$ with $r(a) = r$, called the asymptotic center of $\{x_\alpha : \alpha \in A\}$ in C. A useful lemma is a result proved in [**5**], which we state here as:

LEMMA 1. $r^2 + ||a - x||^2 \leq r(x)^2$ *for every* $x \in C$.

As a consequence, we have the following.

LEMMA 2 (LIM [**8**]). *Let $\{y_\beta\}$ be a net of C such that* $\limsup_\beta r(y_\beta) \le r$. *Then $y_\beta \to a$.*

PROOF. By Lemma 1, we have $r^2 + ||a - y_\beta||^2 \le r(y_\beta)^2$ and hence

$$r^2 + \limsup_\beta ||a - y_\beta||^2 \le \limsup_\beta r(y_\beta)^2 \le r^2.$$

Therefore we have $\limsup_\beta ||a - y_\beta||^2 = 0$ and this implies $y_\beta \to a$.

We also know that if $\{x_\alpha\} \subseteq C$ and if $\{x_\alpha\}$ converges weakly to $y \in C$ then $y = a$ [**2**, Theorem 4.2]. Let Q be the metric projection of H onto $F(\mathscr{S})$. Then, by Phelps [**10**], Q is nonexpansive. Before proving a nonlinear ergodic theorem, we prove the following crucial lemma.

LEMMA 3. *Let C be a closed convex subset of a real Hilbert space H and let S be a right reversible semitopological semigroup. Let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C with* $\limsup_s k_s \le 1$. *Suppose that $F(\mathscr{S}) \ne \varnothing$. Then for each $x \in C$, $\{QT_s x : s \in S\}$ converges to the asymptotic center of $\{T_s x : s \in S\}$ in $F(\mathscr{S})$.*

PROOF. Let z be the asymptotic center of $\{T_s x : s \in S\}$ in $F(\mathscr{S})$. Then, for all $s, t \in S$ we obtain

$$\begin{aligned} r(QT_s x) &\le \sup_{a \ge ts} ||T_a x - QT_s x|| = \sup_{a \ge t} ||T_{as} x - QT_s x|| \\ &= \sup_{a \ge t} ||T_a T_s x - T_a Q T_s x|| \le \left(\sup_{a \ge t} k_a\right) ||T_s x - QT_s x|| \\ &\le \left(\sup_{a \ge t} k_a\right) ||T_s x - z|| \end{aligned}$$

and hence

$$r(QT_s x) \le \limsup_t k_t ||T_s x - z|| \le ||T_s x - z||$$

for all $s \in S$. Therefore we have $\limsup_s r(QT_s x) \le \limsup_s ||T_s x - z|| \le r$. By Lemma 2, we obtain $QT_s x \to z$.

THEOREM 2. *Let C be a closed convex subset of a Hilbert space H and let S be a right reversible semitopological semigroup. Let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C with* $\limsup_s k_s \le 1$. *Suppose that*

$$F(\mathscr{S}) = \bigcap \{F(T_s) : s \in S\} \ne \varnothing.$$

Then the following are equivalent:

(a) $\bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \ge s\} \cap F(\mathscr{S}) \ne \varnothing$ *for each $x \in C$.*

(b) *There is a nonexpansive retraction P of C onto $F(\mathscr{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{\mathrm{co}}\{T_t x : t \in S\}$ for every $x \in C$.*

PROOF. (b)$\Rightarrow$(a). Let $x \in C$. Then $Px \in F(\mathscr{S})$. Also

$$Px \in \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \ge s\}.$$

In fact,

$$Px = PT_s x \in \overline{\mathrm{co}}\{T_t T_s x : t \in S\} \subset \overline{\mathrm{co}}\{T_t x : t \ge s\}$$

for every $s \in S$.

(a)⇒(b). Let $x \in C$ and $f \in F(\mathscr{S})$. Then for each $s, t \in S$, we have

$$\limsup_a ||T_a x - f|| \le \sup_{a \ge ts} ||T_a x - f|| = \sup_{a \ge t} ||T_{as} x - f||$$
$$= \sup_{a \ge t} ||T_a T_s x - T_a f|| \le \left(\sup_{a \ge t} k_a\right) ||T_s x - f||$$

and hence

$$\limsup_a ||T_a x - f|| \le \left(\limsup_t k_t\right) ||T_s x - f|| \le ||T_s x - f||$$

for every $s \in S$. So $\limsup_s ||T_s x - f|| \le \liminf_s ||T_s x - f||$ and hence the $\lim_s ||T_s x - f||$ exists. Let Q be the metric projection of H onto $F(\mathscr{S})$. Then by Lemma 3, $\{QT_s x\}$ converges to the asymptotic center z of $\{T_s x : s \in S\}$ in $F(\mathscr{S})$. Let $u \in \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \ge s\} \cap F(\mathscr{S})$. Then, since

$$||z - u||^2 = ||T_s x - u||^2 - ||T_s x - z||^2 - 2\langle z - u, T_s x - z\rangle$$

for every $s \in S$, we have

$$||z - u||^2 + 2\lim_s \langle z - u, T_s x - z\rangle = \lim_s ||T_s x - u||^2 - \lim_s ||T_s x - z||^2$$
$$= \limsup_s ||T_s x - u||^2 - \limsup_s ||T_s x - z||^2$$
$$= r(u)^2 - r^2 \ge 0.$$

Let $\varepsilon > 0$. Then we have

$$2\lim_s \langle z - u, T_s x - z\rangle > -||z - u||^2 - \varepsilon.$$

Hence there exists $s_0 \in S$ such that

$$2\langle z - u, T_s x - z\rangle > -||z - u||^2 - \varepsilon$$

for every $s \ge s_0$. Since $u \in \overline{\mathrm{co}}\{T_t x : t \ge s_0\}$, we have

$$2\langle z - u, u - z\rangle \ge -||z - u||^2 - \varepsilon.$$

This inequality implies $||z - u||^2 \le \varepsilon$. Since ε is arbitrary, we have $z = u$. Therefore

$$\bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \ge s\} \cap F(\mathscr{S}) = \{z\}.$$

Set $Px = \lim_t QT_t x$ for every $x \in C$. Then we have $T_s Px = Px$ and

$$PT_s x = \lim_t QT_t T_s x = \lim_t QT_{ts} x = Px$$

for every $s \in S$ and $x \in C$. From $\{Px\} = \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \ge s\} \cap F(\mathscr{S})$, it is obvious that $Px \in \overline{\mathrm{co}}\{T_s x : s \in S\}$ for each $x \in C$. Since

$$||Px - Py|| = \lim_t ||QT_t x - QT_t y|| \le \limsup_t ||T_t x - T_t y||$$
$$\le \left(\limsup_t k_t\right) ||x - y|| \le ||x - y||$$

for every $x, y \in C$, it follows that P is nonexpansive.

We now turn to consider the weak convergence of $\{T_s x : s \in S\}$ and obtain the results similar to these of Lau [**7**] and Passty [**9**]. We denote by $\omega(x)$ the set of all weak limit points of subnets of the net $\{T_s x : s \in S\}$. Note that if $F(\mathscr{S})$ is nonempty then $\{T_s x : s \in S\}$ is bounded and hence $\omega(x)$ is nonempty. We start with proving the following lemma.

LEMMA 4. *Let C be a closed convex subset of a Hilbert space H and let S be a right reversible semitopological semigroup. Let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C with $\limsup_s k_s \leq 1$. Suppose that $F(\mathscr{S}) \neq \emptyset$. Let $x \in C$. If $\omega(x) \subseteq F(\mathscr{S})$, then the net $\{T_s x : s \in S\}$ converges weakly to some $y \in F(\mathscr{S})$.*

PROOF. By Lemma 3, the net $\{QT_s x : s \in S\}$ converges strongly to some $y \in F(\mathscr{S})$. Since $\omega(x) \subseteq \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \geq s\}$, we have

$$\bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \geq s\} \cap F(\mathscr{S}) \supseteq \omega(x) \neq \emptyset.$$

Hence, as in the proof of Theorem 2, we obtain

$$\{y\} = \bigcap_{s \in S} \overline{\mathrm{co}}\{T_t x : t \geq s\} \cap F(\mathscr{S}).$$

Therefore we have $\{y\} = \omega(x)$.

Let C be a closed convex subset of a Hilbert space H and let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C. A subset G of S is called a uniformly generating set of $\mathscr{S}$ if for each $s \in S$ and $\varepsilon > 0$, there exist $g_1, g_2, \ldots, g_m \in G$ such that

$$||T_s x - T_{g_1 g_2 \cdots g_m} x|| < \varepsilon$$

for every $x \in C$.

THEOREM 3. *Let C be a closed convex subset of a Hilbert space H and let S be a right reversible semitopological semigroup. Let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C with $\limsup_s k_s \leq 1$. Suppose that $F(\mathscr{S}) \neq \emptyset$. Let $x \in C$. If $\lim_s ||T_{gs} x - T_s x|| = 0$ for all g in a uniformly generating set G of $\mathscr{S}$, then the net $\{T_s x : s \in S\}$ converges weakly to some $y \in F(\mathscr{S})$.*

PROOF. By Lemma 4, it suffices to show that $\omega(x) \subseteq F(\mathscr{S})$. Let $\{T_{s_\alpha} x\}$ be a subnet of $\{T_s x : s \in S\}$ converging weakly to some $y \in C$. Let $\varepsilon > 0$. Then for each $t \in S$, there exist $g_1, g_2, \ldots, g_m \in G$ such that

$$||T_t T_{s_\alpha} x - T_{g_1 g_2 \cdots g_m} T_{s_\alpha} x|| < \varepsilon$$

for every α. Hence we have

$$\begin{aligned}
||T_{s_\alpha} x - T_t y|| &\leq ||T_{s_\alpha} x - T_{g_1 g_2 \cdots g_m s_\alpha} x|| + ||T_{g_1 g_2 \cdots g_m s_\alpha} x - T_{t s_\alpha} x|| + ||T_{t s_\alpha} x - T_t y|| \\
&\leq ||T_{s_\alpha} x - T_{g_1 g_2 \cdots g_m s_\alpha} x|| + \varepsilon + k_t ||T_{s_\alpha} x - y|| \\
&\leq ||T_{s_\alpha} x - T_{g_1 s_\alpha} x|| + ||T_{g_1 s_\alpha} x - T_{g_1 g_2 \cdots g_m s_\alpha} x|| + \varepsilon + k_t ||T_{s_\alpha} x - y|| \\
&\leq ||T_{s_\alpha} x - T_{g_1 s_\alpha} x|| + k_{g_1} ||T_{s_\alpha} x - T_{g_2 \cdots g_m s_\alpha} x|| + \varepsilon + k_t ||T_{s_\alpha} x - y|| \\
&\leq ||T_{s_\alpha} x - T_{g_1 s_\alpha} x|| + k_{g_1} ||T_{s_\alpha} x - T_{g_2 s_\alpha} x|| + k_{g_1} k_{g_2} ||T_{s_\alpha} x - T_{g_3 s_\alpha} x|| \\
&\quad + \cdots + k_{g_1} k_{g_2} \cdots k_{g_{m-1}} ||T_{s_\alpha} x - T_{g_m s_\alpha} x|| + \varepsilon + k_t ||T_{s_\alpha} x - y||
\end{aligned}$$

and

$$\limsup_\alpha ||T_{s_\alpha} x - T_t y|| \leq \varepsilon + k_t \limsup_\alpha ||T_{s_\alpha} x - y||.$$

Since ε is arbitrary, we have

$$\limsup_\alpha ||T_{s_\alpha} x - T_t y|| \leq k_t \limsup_\alpha ||T_{s_\alpha} x - y||.$$

On the other hand, since y is the asymptotic center of $\{T_{s_\alpha}x\}$ in C we obtain

$$\limsup_t r(T_t y) \leq \left(\limsup_t k_t\right) r(y) \leq r$$

and hence $T_t y \to y$. Therefore we have $y \in F(\mathscr{S})$.

REMARK. Let γ be a positive real number and let $\mathscr{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup with $\limsup_s k_s \leq \gamma$. Then, putting $k'_s = \sup_{t \geq s} k_t$, we have

$$||T_s x - T_s y|| \leq k_s ||x - y|| \leq \sup_{t \geq s} k_t ||x - y|| = k'_s ||x - y||$$

and $\lim_s k'_s = \limsup_s k_s$. Hence $\mathscr{S}$ is a Lipschitzian semigroup with $\lim_s k'_s \leq \gamma$.

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