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L^2 BOUNDEDNESS OF HIGHLY OSCILLATORY INTEGRALS ON PRODUCT DOMAINS

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ABSTRACT. We prove L^2 boundedness of the oscillatory singular integral

$$Tf(x,y) = \iint\limits_{D_y} \frac{\exp(2\pi i N(y)x')}{x'y'} f(x-x', y-y')\, dx'\, dy'$$

where $N(y)$ is an arbitrary integer-valued L^∞ function and where nothing is assumed on the dependency upon y of the domain of integration D_y. We also prove L^2 boundedness of the corresponding maximal opertaor. Operators of this kind appear in a problem of a.e. convergence of double Fourier series.

1. Introduction. Recently the theory of singular integrals on product domains $\mathbf{R}^n \times \mathbf{R}^m$ has received much attention [**1**]–[**5**]. In what follows, for simplicity we shall assume $n = m = 1$.

This theory began by considering operators of the following kind:

$$H_1 f(x,y) = \iint\limits_{D} \frac{1}{x', y'} f(x-x', y-y')\, dx'\, dy' \tag{1}$$

where $D \subseteq \mathbf{R} \times \mathbf{R}$ is symmetric with respect to the origin (more precisely the cut off of the domain of integration given by χ_D was smooth [**4**]) but not necessarily a rectangle, otherwise H_1 would simply be the double Hilbert transform.

The theory went on by showing that operators as in (1) can take domains of integration arbitrarily depending upon one of the two variables x or y and still remain bounded. The operator is then (see [**4, 5**])

$$H_2 f(x,y) = \iint\limits_{D_y} \frac{1}{x'y'} f(x-x', y-y')\, dy'\, dx', \tag{2}$$

where nothing is assumed on the dependency upon y of D_y. One is led to this operator by the study of a problem of almost everywhere convergence of double Foureir series (see [**6**] where the partial sums operator is precisely S_{NN^2}) or, in other words, by the study of the boundedness on $L^p(\mathbf{T} \times \mathbf{T})$ of the following operator

$$\int_{-1}^{1} \int_{-1}^{1} \frac{e^{2\pi i [N(x,y)x' + N^2(x,y)y']}}{x'y'} f(x-x', y-y')\, dx'\, dy',$$

where $N(x,y)$ is an arbitrary integer-valued L^∞-function. For the same problem (see [**6**]) another operator has been used, that is

$$\iint \frac{e^{2\pi i N(y)x'}}{x'y'} f(x-x', y-y')\, dy'\, dx', \tag{3}$$

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where $N(y)$ is an arbitrary integer-valued, L^∞-function and where the domain of integraton is a rectangle. This operator is easily decoded: if one integrates first in y' and then in x', to an exponential factor, (3) is the Hilbert transform in y' followed by the Hillbert transform in x'. Now it is natural to consider the operator

$$(4) \qquad Tf(x,y) = \iint_{D_y} \frac{e^{2\pi i N(y)x'}}{x'y'} f(x-x', y-y')\, dy'\, dx'$$

which has (2) and (3) as particular cases. T is by far a more difficult operator to study than H_2, due to the potentially highly oscillating exponential which, in particular, deprives the kernel of its cancellation property and even of substitutes of it, see [**3**] (condition (1.2) of [**3**], on the other hand, is already not satisfied by H_2). It would be interesting to prove that T is bounded on L^p, $1 < p < \infty$, for this would show that singular integrals on product spaces not only can take "arbitrary" domains of integration, but also "arbitrary" oscillations.

We are going to prove that the operator T is bounded on $L^2(\mathbf{R}\times\mathbf{R})$ with norm independent of $\{D_y\}_y$, of $N(y)$ and of the L^∞-norm of $N(y)$. Moreover we will consider the maximal operator

$$\tilde{T}f(x,y) = \operatorname*{Sup}_{\varepsilon>0} \left| \iint_{D_y;|x'|>\varepsilon} \frac{e^{2\pi i N(y)x'}}{x'y'} f(x-x', y-y')\, dy'\, dx' \right|$$

and prove that $\tilde{T}f(x,y) \le c\{M_{x'}\tilde{H}_{y'}f(x,y) + M_{x'}Tf(x,y)\}$ where c is an absolute constant, $M_{x'}$ denotes the maximal function acting on x' and $\tilde{H}_{y'}$ the maximal Hilbert transform acting on y'. This in particular proves that $\tilde{T}$ is bounded on L^2.

2. Results. Our domains of integration are going to be defined by smooth dyadic cut-offs. Let $\phi(x')$ be an odd, C^∞-function supported on $\{x' \colon \frac{1}{2} \le |x'| \le 2\}$ such that $\sum_{k=-\infty}^{+\infty} 2^k\phi(2^k x') = \sum_k \phi_k(x') = 1/x'$, $x' \ne 0$. Then $1/x'y' = \sum_{k,h=-\infty}^{+\infty} \phi_k(x')\phi_h(y')$, $x' \ne 0$, $y' \ne 0$. Let $B_y \subseteq \mathbf{Z}\times\mathbf{Z}$ have the following property: for every $k \in \mathbf{Z}$ and $y \in \mathbf{R}$ fixed there exists $\delta(k,y) \ge 0$ such that

$$B_y \cap \left\{ \bigcup_h (k,h) \right\} = \{(k,h) \colon 2^{-h} \le \delta(k,y)\}.$$

Let $N(y)$ be an integer-valued L^∞-function. Then consider the operator

$$\begin{aligned} Tf(x,y) &= \sum_{(k,h)\in B_y} e^{2\pi i N(y)x'} \phi_k\phi_h * f(x,y) \\ &= \int \sum_{k=-\infty}^{+\infty} e^{2\pi i N(y)x'} \phi_k(x') \int \sum_{2^{-h}\le\delta(k,y)} \phi_h(y') f(x-x', y-y')\, dy'\, dx', \end{aligned}$$

defined in principal value sense. The existence of the limit for the operator

$$\sum_{(k,h)\in B_y} \phi_k(x')\phi_h(y') * f(x,y)$$

has been discussed in [**4**] and [**5**]. Since $N(y)$ takes on only a finite number of values the same holds for our operator T. We are going to prove the following theorem.

THEOREM 1. *Under the assumptions stated above there exists a constant A independent of f, $\{B_y\}_y$, $N(y)$ and of the L^∞-norm of $N(y)$ such that $\|Tf\|_2 \leq A\|f\|_2$.*

PROOF. One wants to break up the binding between the integration in y' and x', then switch the order of integration (this can be done only after eliminating $N(y)$) so to be able to control the operator in y' by the maximal Hilbert transform. This is obtained by moving on the Fourier transform side of the x-variable and by a repeated application of Plancherel theorem. In the process we are going to use the following inequality $\sum_{k=-\infty}^{+\infty} |\hat{\phi}_k(\xi)| \leq c$, which is easily checked. With y fixed we have that

$$\begin{aligned}
\int |Tf(x,y)|^2\,dx &= \int |\widehat{Tf}(\xi,y)|^2\,d\xi \\
&= \int \left| \sum_k \hat{\phi}_k(\xi - N(y)) \left(\sum_{2^{-h}<\delta(k,y)} \phi_h(y') * f(\cdot,y) \right)^{\wedge} (\xi) \right|^2 d\xi \\
&\leq \int \sum_k |\hat{\phi}_k(\xi - N(y))| \operatorname*{Sup}_{h_0} \left| \left(\sum_{h \geq h_0} \phi_h(y') * \hat{f}(\xi,y') \right)(y) \right|^2 d\xi \\
&\leq c \int |\tilde{H}_{y'} \hat{f}(\xi,y')(y)|^2\,d\xi.
\end{aligned}$$

Here and in what follows $\tilde{H}_{y'}$ denotes a smooth variant of the maximal Hilbert transform, namely $\tilde{H}_{y'}g(y) = \operatorname{Sup}_{h_0} |\sum_{h\geq h_0} \phi_h * g(y)|$ which has been studied in [**4**]. Then by switching the order of integration we have that

$$\begin{aligned}
\iint |Tf(x,y)|^2\,dx\,dy &\leq c \iint |\tilde{H}_{y'} \hat{f}(\xi,y')(y)|^2\,dy\,d\xi \\
&\leq A \iint |\hat{f}(\xi,y')|^2\,dy'\,d\xi \leq A \iint |f(x',y')|^2\,dx'\,dy'.
\end{aligned}$$

This proves the theorem.

Now we consider the maximal operator

$$\tilde{T}f(x,y) = \operatorname{Sup} \left| \sum_{\substack{(k,h)\in B_y \\ k\leq k_0}} e^{2\pi i N(y)x'} \phi_k(x')\phi_h(y') * f(x,y) \right|$$

and we prove the following

THEOREM 2. *Under the same assumptions of Theorem* 1 *there exists a constant c independent of f, $\{B_y\}_y$, $N(y)$ and the L^∞-norm of $N(y)$ such that $\tilde{T}f(x,y)$ defined above satisfies the inequality*

$$\tilde{T}f(x,y) \leq c\{M_{x'}\tilde{H}_{y'}f(x,y) + M_{x'}Tf(x,y)\}.$$

In particular $\|\tilde{T}f\|_2 \leq c\|f\|_2$.

PROOF. In this case we are going to move the exponential toward the function f and write

$$\tilde{T}f(x,y) = \operatorname*{Sup}_{k_0} \left| \int \sum_{k \le k_0} \phi_k(x') e^{2\pi i N(y)(x-x')} F_k(x-x',y)\, dx' \right|$$

where

$$F_k(x',y) = \int \sum_{2^{-h} \le \delta(k,y)} \phi_h(y') f(x', y-y')\, dy'.$$

Clearly $|F_k(x',y)| \le \tilde{H}_{y'} f(x',y)$. Let $\theta(x')$ be a positive, decreasing, C^∞-function supported on $\{|x'| \le 1\}$ and such that $\int_{-1}^{1} \theta(x')\, dx' = 1$. We write $\theta_k(x') = 2^k \theta(2^k x')$. We are going to prove the inequality

(5)

$$\begin{aligned} &\left| \int \sum_{k \le k_0} \phi_k(x') e^{2\pi i N(y)(x-x')} F_k(x-x',y)\, dx' \right. \\ &\quad \left. - \int \sum_{k=-\infty}^{\infty} \left(\int \theta_{k_0}(x'') \phi_k(x'-x'')\, dx'' \right) e^{2\pi i N(y)(x-x')} F_k(x-x',y)\, dx' \right| \\ &\quad = |\mathrm{I} - \mathrm{II}| \le c M_{x'} \tilde{H}_{y'} f(x,y). \end{aligned}$$

If $|x'| < 10 2^{-k_0}$ then

$$\begin{aligned} |\mathrm{I}| &\le \int_{|x'| < 10 2^{-k_0}} \sum_{k \le k_0} |\phi_k(x')|\, |F_k(x-x',y)|\, dx' \\ &\le c \int_{|x'| < 10 2^{-k_0}} 2^{k_0} \tilde{H}_{y'} f(x-x',y)\, dx' \le c M_{x'} \tilde{H}_{y'} f(x,y) \end{aligned}$$

and

$$\begin{aligned} |\mathrm{II}| &\le \int_{|x'| < 10 2^{-k_0}} \sum_k \left| \int \theta_{k_o}(x'') \phi_k(x'-x'')\, dx'' \right| |F_k(x-x',y)|\, dx' \\ &\le c M_{x'} \tilde{H}_{y'} f(x,y) \end{aligned}$$

since $\|\theta_{k_0} * \sum_k \phi_k\|_\infty \le c 2^{k_0}$ (see [**4**] and [**5**]). If $|x'| \ge 10 2^{-k_0}$ then $\sum_k \theta_{k_0} * \phi_k(x') = \sum_{k \le k_0} \theta_{k_0} * \phi_k(x')$ and so

$$\begin{aligned} |\mathrm{I} - \mathrm{II}| &= \left| \int \left(\int \sum_{k \le k_0} (\phi_k(x') - \phi_k(x'-x'')) \phi_{k_0}(x'')\, dx'' \right) \right. \\ &\qquad\qquad \left. e^{2\pi i N(y)(x-x')} F_k(x-x',y)\, dx' \right| \\ &\le \iint \sum_{k \le k_0} |\phi_k(x') - \phi_k(x'-x'')| \theta_{k_0}(x'')\, dx'' |F_k(x-x',y)|\, dx' \\ &\le c \int_{|x'| \ge 10 2^{-k_0}} \frac{2^{-k_0}}{(x')^2} \tilde{H}_{y'} f(x-x',y)\, dx' \le c M_{x'} \tilde{H}_{y'} f(x,y). \end{aligned}$$

This proves estimate (5). Now by a limiting argument one can see that

$$\sum_k (\theta_{k_0} * \phi_k)(x') * e^{2\pi i N(y)x'} F_k(x', y)(x)$$
$$= \theta_{k_0} * \left(\sum_k \phi_k(x') * e^{2\pi i N(y)x'} F_k(x', y) \right)(x).$$

Since the right-hand side is dominated by $M_{x'} Tf(x, y)$ the theorem is proved.

By the same method and Lemma 2 of [**5**] these results can be extended to $\mathbf{R}^n \times \mathbf{R}^m$, $n, m \geq 1$. We leave this to the interested reader.

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