OSCILLATORY SOLUTIONS FOR CERTAIN DELAY-DIFFERENTIAL EQUATIONS
GEORGE SEIFERT
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ABSTRACT. The existence of oscillatory solutions for a certain class of scalar first order delay-differential equations is proved. An application to a delay logistic equation arising in certain models for population variation of a single species in a constant environment with limited resources for growth is considered.

It is known (cf. [1, 2]) that all solutions of the delay logistic equations

\[ N'(t) = N(t)(a - bN(t) - N(t-1)), \quad t > 0, \]

with \( N(t) = N_0(t) > 0, -1 < t < 0, \) \( N_0 \) continuous, and \( a \) and \( b \) positive constants, satisfy \( N(t) \to a/(b+1) \) as \( t \to \infty \) whenever \( b > 1 \). In [3] it was shown that for any \( b > 0 \), there exists \( a(b) > 0 \) and that if \( 0 < a < a(b) \), there exist solutions \( N(t) \) of (1) which do not oscillate about the equilibrium \( N = a/(b+1) \); in particular, such that, \( N(t) > a/(b+1) \) for \( t \geq 0 \). It is the purpose of this paper to show that for this same \( a(b) \), if \( a > a(b) \), there exist oscillatory solutions about this equilibrium solution. In case \( b < 1 \), this is known; in fact, a Hopf bifurcation (cf. [1]) shows the existence for certain \( a \) of nonconstant positive periodic solutions. However, if \( b > 1 \), the fact that some solutions of (1) approach \( a/(b+1) \) in an oscillatory fashion seems to be new.

The above mentioned result for (1) will follow from a result for a more general scalar delay-differential equation of the form

\[ y'(t) = L(y_t) + N(t, y_t), \quad t > 0. \]

Here \( y_t = y(t+\theta), -1 \leq \theta \leq 0, \) and we assume

\( (H_1) \) \( L(\phi) \) is continuous and linear on \( C = C([-1, 0], R) \) and \( N(t, \phi) \) is continuous on \( R \times C \) and satisfies

\[ |N(t, \phi)| \leq M(t)||\phi||^2, \quad \phi \in C, \ ||\phi|| \leq B_0, \ t \geq 0; \]

where the norm in \( C \) is defined by \( ||\phi|| = \sup\{||\phi(\theta)||: -1 \leq \theta \leq 0\} \), and \( \int_{-\infty}^{\infty} M(t) \, dt < \infty; \)

\( (H_2) \) The characteristic equation for

\[ y'(t) = L(y_t) \]

has a pair of simple pure imaginary roots \( \pm i\beta, \beta > 0, \) and all other roots have negative real parts.
Remark 1. Under assumption (H2), there exists a nonconstant periodic solution \( y^*(t) \) of (3) and positive numbers \( \rho^* \) and \( B, \rho^* < 1, B < B_0/2 \), such that

\[
\max\{y^*(t) : t \in R\} \geq \rho^*, \quad \min\{y^*(t) : t \in R\} \leq -\rho^*,
\]
\[
|y^*(t)| \leq B, \quad t \in R.
\]

This follows from standard theory for solutions of (3); cf., for example, Hale’s monograph [4].

Definition. The real-valued function \( f(t) \) on \([0, \infty)\) is oscillatory if there exist \( t_n \to \infty \) as \( n \to \infty, t_{n+1} > t_n \), such that \((-1)^n f(t_n) > 0, n = 1, 2, \ldots\).

Remark 2. If \( f(t) \) is continuous and oscillatory in this sense, clearly \( f \) must have an unbounded sequence of zeros and cannot be identically zero on any half infinite interval \([t_0, \infty), t_0 \geq 0\).

Theorem 1. If (H1) and (H2) hold, there exists \( \delta_0 > 0 \) such that for each \( \delta, 0 < \delta < \delta_0 \), (2) has an oscillatory solution \( y = w(t) \) such that \( |w(t)| \leq \delta, t \geq 0 \).

Proof. Let \( u(t) \) be the fundamental solution for (3); i.e., let \( u(t) \) solve (cf. appendix)

\[
u'(t) = L(u(t)), \quad t > 0,
\]
\[
u(0) = 1,
\]
\[
u(t) = 0, \quad -1 < t < 0.
\]

From (H2) it follows that there exists \( K > 0 \) such that \( |u(t)| \leq K, t \geq 0 \); again cf. [4, Chapter 7]. For this \( K \), and \( \rho^* \) and \( B \) as in Remark 1, fix \( \varepsilon > 0 \) such that

\[
\varepsilon \int_0^\infty M(t) dt < \frac{\rho^*}{8B^2 K} < (4BK)^{-1};
\]

note that \( \rho^* \leq B \).

Let \( X(B) \) denote the set of real functions \( z \) continuous on \([-1, \infty)\) such that \( z(t) = y^*(t), -1 \leq t \leq 0 \), where \( y^*(t) \) is the periodic solution of (3) as described in Remark 1, and \( |z(t)| \leq 2B, t \geq 0 \). With the topology of uniform convergence on compact subsets of \([-1, \infty)\), the set \( X \) of all real functions continuous on \([-1, \infty)\) is a locally convex linear topological space over the reals, and clearly \( X(B) \subset X \).

Define the map \( T \) on \( X(B) \) to \( X \) by

\[
(Tz)(t) = y^*(t) + \frac{1}{\varepsilon} \int_0^t u(t-s)N(s, \varepsilon z(s)) ds, \quad t > 0,
\]
\[
= y^*(t), \quad -1 \leq t \leq 0,
\]

for any \( z \in X(B) \).

Using (5) with (H1) and the boundedness property of \( u(t) \), we have

\[
(Tz)(t) \leq B + 4KB^2 \varepsilon \int_0^t M(s) ds \leq 2B, \quad t \geq 0;
\]

therefore \( Tz \in X(B) \).

Using (6) and the properties of \( u(t) \) given in (4) it follows that

\[
\frac{d}{dt}(Tz)(t) = y^{**}(t) + \frac{1}{\varepsilon} N(t, \varepsilon z(t)) + \frac{1}{\varepsilon} \int_0^t u'(t-s)N(s, \varepsilon z(s)) ds
\]
\[
= y^{**}(t) + \frac{1}{\varepsilon} N(t, \varepsilon z(t)) + \frac{1}{\varepsilon} \int_0^t L(u_{t-s})N(s, \varepsilon z(s)) ds.
\]
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So since $L$ is bounded, $z \in X(B)$, $u(t)$ is bounded for $t > 0$, and $N$ satisfies the conditions in $(H_1)$, it follows that there exists a constant $C(\varepsilon)$ and that

$$\frac{d}{dt}(Tz)(t) \leq C(\varepsilon), \quad t > 0.$$ 

By a standard argument using the Ascoli-Arzela theorem, it then follows that $TX(B)$ is precompact in the topology of $X$, and by the Schauder-Tychonov fixed point theorem, there exists a $z^* \in X(B)$ such that

$$z^*(t) = y^*(t) + \frac{1}{\varepsilon} \int_0^t u(t-s)N(s,\varepsilon z^*(s)) \, ds, \quad t > 0,$$

$$z^*(t) = y^*(t), \quad -1 \leq t \leq 0.$$ 

Since $u(t)$ is a fundamental solution for (3), it follows that $z^*(t)$ solves

$$z'(t) = L(z_t) + \frac{1}{\varepsilon} N(t,\varepsilon z_t), \quad t > 0,$$

$$z(t) = y^*(t), \quad -1 \leq t \leq 0,$$

and so $y(t) = \varepsilon z^*(t)$ solves (2) for $t > 0$ with $y(t) = \varepsilon y^*(t)$ for $-1 \leq t \leq 0$.

If

$$R_0(t) = \frac{1}{\varepsilon} \int_0^t u(t-s)N(s,\varepsilon z^*(s)) \, ds, \quad t \geq 0,$$

then $z^*(t) = y^*(t) + R_0(t)$, $t \geq 0$, and using the properties of $u$ and $N$ and the fact that $z^* \in X(B)$ it follows that

$$|R_0(t)| \leq 4B^2 K\varepsilon \int_0^t M(s) \, ds \leq \frac{\rho^*}{2}, \quad t \geq 0.$$ 

So

$$|z^*(t) - y^*(t)| \leq \rho^* / 2, \quad t \geq 0.$$ 

But using the properties of $y^*(t)$ mentioned in Remark 1, there exists $t_n \to \infty$ as $n \to \infty$, $t_{n+1} > t_n$, such that $y^*(t_n) \geq \rho^*$, $n = 1, 2, \ldots$. Using (8) it follows easily that

$$z^*(t_n) \geq \rho^*/2, \quad n = 1, 2, \ldots.$$ 

Similarly, there exists a sequence $\tau_n \to \infty$ as $n \to \infty$, $\tau_{n+1} > \tau_n$, such that $y^*(\tau_n) \leq -\rho^*$, and so

$$z^*(\tau_n) \leq -\rho^*/2, \quad n = 1, 2, \ldots.$$ 

Thus the solution $y(t) = \varepsilon z^*(t) \equiv w(t)$ of (2) is oscillatory. Now define $\varepsilon_0$ to be the supremum of the set of all $\varepsilon > 0$ for which this argument holds. Since for such $\varepsilon > 0$, $|w(t)| \leq \varepsilon B$, with $B$ as in Remark 1, and if we take $\delta_0 = 2\varepsilon_0 B$, our theorem is proved. Note that from (5), $\varepsilon_0 \leq (4BK M)^{-1}$, where $M = \int_0^\infty M(t) \, dt$.

REMARK 3. If $\beta$ is as in $(H_2)$, it can be shown that the $t_n$ and $\tau_n$ in our proof above can be chosen such that

$$t_{n+1} - t_n \leq 2\pi / \beta, \quad \text{and} \quad \tau_{n+1} - \tau_n \leq 2\pi / \beta.$$ 

This follows because $y^*(t)$ can be chosen to be a linear combination of $\sin \beta t$ and $\cos \beta t$. We omit the details.
We now return to the delay logistic equation (1) with \( b > 1 \). If we make the change of variables \( x(t) = N(t) - \frac{a}{b+1} \) (1) becomes
\[
x'(t) = -(\frac{a}{b+1} + x(t))(bx(t) + x(t - 1)),
\]
and the linear part of (9) is the equation
\[
x'(t) = -(\frac{a}{b+1})(bx(t) + x(t - 1)).
\]
It is not difficult to see that all roots of the characteristic equation for (10) have negative real part, cf. [5]. From a result in [3], it also follows that if
\[
a > (b+1)/m(b),
\]
where \( m(b) \) is the unique root of \( b = m \log m - 1 \), then all roots of this characteristic equation are nonreal. A direct examination of this characteristic equation also shows that all nonreal roots must be simple.

Under the change of variable \( y(t) = x(t) \exp(\mu t) \), where \( \mu \) is a real constant, (9) becomes
\[
y'(t) = A(\mu)y(t) + B(\mu)y(t - 1) + f(y(t), y(t - 1)) \exp(-\mu t)
\]
where \( A(\mu) = \mu - ab/(b+1) \), \( B(\mu) = -ae^\mu/(b+1) \), and
\[
f(y,z) = -(by^2 + yze^\mu).\]
It is easy to see that if \( \alpha \) is the real part of a root of the characteristic equation for (9), then \( \mu + \alpha \) is the real part of a corresponding root of the characteristic equation for the linear part of (12), namely
\[
y'(t) = A(\mu)y(t) + B(\mu)y(t - 1).
\]
So if we choose \( \mu = -\max\{\text{Re} \lambda: \lambda \text{ is a root of the characteristic equation for (10)}\} \), then the characteristic equation for (13) has pure imaginary roots \( \pm i\beta, \beta > 0 \), which are simple if (11) holds. Also all other roots of this equation for (13) have negative real parts. Clearly \( \mu > 0 \). So we see that all the hypotheses of Theorem 1 are satisfied for (11) and we have the following.

**Theorem 2.** If \( b > 1 \) and \( a > (b+1)/m(b) \), where \( m(b) \) is as defined above, then there exist oscillatory solutions of (9) of arbitrarily small amplitude; i.e., there exist solutions of (1) which oscillate about \( \frac{a}{b+1} \).

The proof of this theorem now follows easily, since by Theorem 1, there exist such oscillatory solutions \( y(t) \) of (12) and so the corresponding solutions \( x(t) = y(t) \exp(-\mu t) \) are also oscillatory.

An open question presents itself: under the hypotheses of Theorem 2, are all solutions of (9) oscillatory?

**Appendix.** In the strict sense, the initial function on \([-1,0]\) for the equation defining \( u(t) \) in (4) is not in \( C \). What is really involved here (a point not entirely clear in [4]) is that \( u(t) \) solves the initial value problem
\[
u'(t) = \begin{cases} 
0 & 0 \leq t < 1, \\
\int_{-t}^{0} u(t+s) \, d\eta(s), & t \geq 1, 
\end{cases}
\]
and
\[
u(0) = 1.
\]
where $\eta(s)$ is a function of bounded variation which by the Riesz representation theorem characterizes $L$; i.e. is such that $L(\phi) = \int_{-1}^{0} \phi(s) \, d\eta(s)$ for $\phi \in C$. This initial value problem can be shown to have a solution in a fairly standard way such as by the method of successive approximations.

REFERENCES