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HUNT'S HYPOTHESIS FOR LEVY PROCESSES

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ABSTRACT. In this note, a general condition is given implying the validity of Hunt's hypothesis (H) for Levy processes in d-dimensions.

Introduction. It is of interest to some probabilists to know when a Levy process satisfies Hunt's Hypothesis (H), namely, when semipolar sets are indeed polar. Getoor conjectured some twenty years ago that nearly all Levy processes have this property. Let us describe some of the known results. Let ψ denote the exponent of the Levy process in d-dimensions. If $d = 1$ Kesten showed [**7**] that one point sets are nonpolar if and only if

$$\int_0^\infty \operatorname{Re}([1+\psi(\alpha)]^{-1})\, d\alpha < \infty.$$

Port and Stone showed [**8**] that for the asymmetric Cauchy process on the line every point is regular for itself.

These considerations received a big push when Kanda [**5**] and Forst [**3**] independently showed that for $d > 1$ all stable processes of index $\alpha \neq 1$ satisfy Hunt's Hypothesis (H). These methods did not settle the case $\alpha = 1$. Following a different line of reasoning and using the result of Port and Stone mentioned above, Kanda [**6**] settled this problem even for this case assuming the linear term vanishes. In a completely elementary fashion it is shown in [**4**] that α-subordinates of general Hunt processes satisfy (H). In this note we give a sufficient condition for the validity of (H). This condition easily includes the above results and goes farther. For example it is not necessary to assume that the linear term vanishes as Kanda did in [**6**]. As another example our result easily implies that the independent product of one dimensional Cauchy processes satisfies (H). As far as we know this is a new result. From the proof it will seem that the condition is not far from being necessary.

The main result. In the following X will denote a Levy process on R^d with exponent ψ. Namely for $\alpha \in R^d$,

$$E[\exp(2\pi i\alpha X_t)] = \exp(-t\psi) \tag{1}$$

holds. We will write

$$A = 1 + \operatorname{Re}\,\psi, \qquad B = |1+\psi|. \tag{2}$$

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We assume that all 1-excessive functions for X are lower semicontinuous. Denote the 1-potential kernel of X by u^1:

$$\int u^1(y-x)f(y)\,dy = E^x\left[\int_0^\infty e^{-t}f(X_t)\,dt\right]. \tag{3}$$

Our assumptions regarding lower semicontinuity of all excessive functions guarantees that such u^1 exists. Indeed if $f \geq 0$ and Borel

$$\int U^1 f(x)\,dx = \int f(x)\,dx.$$

So if $f = 0$ a.e. then $U^1 f(x) = 0$ a.e. and hence everywhere by lower semicontinuity. Now one appeals to Theorem 1.4 of [**1**, p. 254] to get densities u^1 with nice properties. If μ is a positive Radon measure its potential $U^1\mu$ is

$$U^1\mu(x) = \int u^1(y-x)\ \mu(dx).$$

$U^1\mu$ is called regular if $U^1\mu(X)$ is continuous wherever X is continuous. Hypothesis (H) of Hunt holds if and only if $U^1\mu$ is regular for all μ with compact support such that $U^1\mu$ is bounded [**1**, Propositions (4.8) and (4.9), p. 288].

Let μ be a finite measure. Denote by $\hat{\mu}$ its Fourier transform

$$\hat{\mu}(\alpha) = \int \exp(2\pi i\alpha x)\ \mu(dx).$$

We say that μ has finite 1-energy if $\int B^{-2}A|\hat{\mu}|^2 < \infty$.

REMARK. If μ is a finite measure such that $U^1\mu$ is bounded then μ has finite 1-energy. Indeed then the function

$$\int [u^1(y-z-x) + u^1(z-y+x)]\,\mu(dy)\,\mu(dz)$$

is bounded and integrable. Its Fourier transform equals $2AB^{-2}|\hat{\mu}|^2$ which is nonnegative. Hence it is integrable by the corollary on p. 482 of [**2**]. The following Theorem is a simple consequence of Lemma 2.1 and Theorem 1.5 of [**9**].

THEOREM 1. *Let μ be a finite measure of finite 1-energy i.e. $\int B^{-2}A|\hat{\mu}|^2 < \infty$. Then*

$$\lim_{\lambda\to\infty} \int |\hat{\mu}|^2(\lambda + \operatorname{Re}\psi)|\lambda+\psi|^{-2} \tag{4}$$

exists. The limit is zero if and only if $U^1\mu$ is regular.

With these preliminaries aside we come to the main result of this note.

THEOREM 2. *Suppose there is an increasing function f on $[1,\infty)$ such that*

$$\int_N^\infty (\lambda f(\lambda))^{-1}\,d\lambda = \infty \tag{5}$$

for every N. Suppose further that with A and B as in (2) $B \leq Af(A)$. Then Hypothesis (H) of Hunt holds.

PROOF. By the above Remark, Propositions (4.8) and (4.9), p. 288 of [**1**] it is sufficient to show that $U^1\mu$ is regular whenever μ is a finite measure and has finite 1-energy.

Let μ be a finite measure of finite 1-energy (as defined in the statement of Theorem 1). Now for $\lambda \geq 1$,

$$|\lambda + \psi|^{-2} A \leq B^{-2} A.$$

Hence by dominated convergence theorem the limit in (4) is equal to

$$\lim_{\lambda \to \infty} \int |\lambda + \psi|^{-2} \lambda |\hat{\mu}|^2. \tag{6}$$

Again for $\lambda \geq 1$, it is elementary to see that

$$\left| \frac{\lambda}{|\lambda + \psi|^2} - \frac{\lambda}{\lambda^2 + B^2} \right| \leq \frac{2A}{B^2}.$$

And again by dominated convergence the limit in (6) is equal to

$$\lim_{\lambda \to \infty} \int \lambda (\lambda^2 + B^2)^{-1} |\hat{\mu}|^2. \tag{7}$$

We need one more reduction. Now

$$\int \lambda (\lambda^2 + B^2)^{-1} 1_{\lambda \leq A} |\hat{\mu}|^2$$

tends to zero by dominated convergence because

$$\frac{\lambda}{\lambda^2 + B^2} 1_{\lambda \leq A} \leq \frac{A}{B^2}$$

Thus the limit

$$\lim_{\lambda \to \infty} \int (\lambda^2 + B^2)^{-1} \lambda 1_{A \leq \lambda} |\hat{\mu}|^2 \tag{8}$$

exists and equals the limit in (4). If the limit in (8) is positive, then the integral of $\int \lambda(\lambda^2 + B^2)^{-1} 1_{A \leq \lambda} |\hat{\mu}|^2$ relative to the infinite measure $(\lambda f(\lambda))^{-1}\, d\lambda$ must be infinite. However

$$\begin{aligned}
\int_1^\infty \lambda^{-1} f(\lambda)^{-1}\, d\lambda \int \lambda(\lambda^2 + B^2)^{-1} |\hat{\mu}|^2 1_{A \leq \lambda} \\
= \int |\hat{\mu}|^2 \int_A^\infty [f(\lambda)(\lambda^2 + B^2)]^{-1} \\
\leq \pi 2^{-1} \int [B f(A)]^{-1} |\hat{\mu}|^2 \\
\leq \pi 2^{-1} \int A B^{-2} |\hat{\mu}|^2 < \infty.
\end{aligned}$$

because $B \leq A f(A)$. This proves that the limit in (4) is zero and hence Hunt's Hypothesis (H) holds.

The exponent ψ of the nondegenerate asymmetric Cauchy process on R^d has the form

$$\psi(z) = i(a, z) + \int_S w(z, \theta)\, \nu(d\theta)$$

where ν is a finite measure on the unit hypersphere S in R^d, $a \in R^d$ and,

$$w(z, \theta) = |(z, \theta)| + 2i\pi^{-1}(z, \theta) \log |(z, \theta)|.$$

Nondegeneracy means that the support of ν is not contained in a hyperplane. In particular $\operatorname{Re}\psi$ is bounded below on the unit sphere. The following simple estimate thus obtains

$$B \leq MA \log A$$

where $A = 1 + \operatorname{Re}\psi$, and $B = |1 + \psi|$. Hence the function $f(\lambda) = M \log \lambda$ meets the requirements of Theorem 2 and we get

COROLLARY 3. *Nondegenerate asymmetric Cauchy processes satisfy* (H).

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