STABLE MAPS INTO THE HILBERT CUBE

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ABSTRACT. A map into the Hilbert cube is stable if each composition with projection onto a finite number of factors is stable. We prove that a map from a compact metric space into the Hilbert cube is stable if and only if it is universal. As a consequence, the composition of a stable map with any self homeomorphism of the Hilbert cube is also stable.

1. Introduction. All spaces are separable metric spaces. A map \( f: X \to \mathbb{I}^n \) is said to be stable if there does not exist a map \( g: X \to S^{n-1} \) with \( f|_{f^{-1}(S^{n-1})} = g|_{f^{-1}(S^{n-1})} \). Stable maps are also known as Alexander-Hopf essential maps [N, GT]. Krasinkiewicz has given a general definition of essential maps into the product of manifolds [K] that coincides with the definition of stable maps in the cases under consideration. It is well known that a space has covering dimension greater than or equal to \( n \) if and only if it admits a stable map into \( \mathbb{I}^n \) [HW]. Note that if \( f: X \to \mathbb{I}^n \) is a stable map and \( h \) is any self-homeomorphism of \( \mathbb{I}^n \), then the composition \( h \circ f \) is also stable, since \( h(S^{n-1}) = S^{n-1} \).

Let \( \mathbb{I}^\infty = \prod_{i=1}^{\infty} [-1, 1] \) denote the Hilbert cube, and for each \( n \) let \( p_n: \mathbb{I}^\infty \to \mathbb{I}^n \) be the projection map onto the first \( n \) factors. The subsets \( A_n = \{(x_i) \in \mathbb{I}^\infty | x_n = -1\} \) and \( B_n = \{(x_i) \in \mathbb{I}^\infty | x_n = 1\} \) are referred to as faces of the Hilbert cube. A map \( f: X \to \mathbb{I}^\infty \) from a compact metric space into the Hilbert cube is said to be stable if the composition \( p_n \circ f: X \to \mathbb{I}^n \) is stable for each \( n \). See [W and B] for a more detailed description of stable maps. In particular, Walsh showed that a map \( f: X \to \mathbb{I}^\infty \) from a compact metric space is stable if and only if the collection of pairs \( \{(f^{-1}(A_i), f^{-1}(B_i)) | i = 1, 2, \ldots \} \) is an essential family for \( X \), i.e., for any sequence of separators \( \{S_i\} \) of \( f^{-1}(A_i) \) and \( f^{-1}(B_i) \), \( \bigcap_{i=1}^{\infty} S_i \neq \emptyset \). It follows that a compact metric space admits a stable map into the Hilbert cube if and only if it is strongly infinite dimensional.

Our goal is to prove a result for stable maps into the Hilbert cube which is analogous to the result noted above for stable maps into \( n \)-cells. Namely, we show that if \( f: X \to \mathbb{I}^\infty \) is stable and \( h \) is any self-homeomorphism of \( \mathbb{I}^\infty \), then the composition \( h \circ f \) is also stable. This gives a partial answer to a question of J. Krasinkiewicz [K, Problem 1]. What we need for the proof is a characterization of stable maps in terms of a property preserved by self-homeomorphisms. Universality, a concept introduced by Holsztyński [H1] and widely used in the study of fixed point theory (see [H2, H3, H4, H5, H6, H7, GT and N]), is such a property. A map
$f: X \to Y$ is said to be universal if for every map $g: X \to Y$ there exists a point $p$ in $X$ with $f(p) = g(p)$. We will show that a map from a compact metric space into the Hilbert cube is stable if and only if it is universal. Then the desired result on preservation of stability by compositions with self-homeomorphisms of $I^\infty$ is an immediate corollary.

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2. Universal maps and stable maps. The following result is contained in [GT, N] and implicitly in [H]. For completeness, we include a short proof.

**THEOREM 1.** Let $n$ be in $\mathbb{Z}^+$. A map $f: X \to I^n$ is stable if and only if it is universal.

**PROOF.** Suppose that $f: X \to I^n$ is a stable map. If $f$ were not universal, then we could find a map $g: X \to I^n$ so that $g(p) \neq f(p)$ for every point $p$ in $X$. Consider $S^{n-1}$ as the boundary of $I^n$ in the usual manner, and define $h: X \to S^{n-1}$ by setting $h(p)$ equal to the intersection of the ray containing $f(p)$ which emanates from $g(p)$ and $S^{n-1}$. Clearly $h$ is continuous and agrees with $f$ on $f^{-1}(S^{n-1})$, contradicting the stability of $f$. Therefore, $f$ must be universal.

For the converse, suppose that $f: X \to I^n$ is universal and again consider $S^{n-1}$ as the boundary of $I^n$. If $f$ were not stable, then we could find a map $g: X \to S^{n-1}$ which agrees with $f$ on $f^{-1}(S^{n-1})$. Composing with the antipodal map $\alpha: S^{n-1} \to S^{n-1}$ would then give a map $\alpha \circ g: X \to S^{n-1} \subset I^n$ so that $\alpha \circ g(p) \neq f(p)$ for any point $p$ in $X$, contradicting the universality of $f$. Thus, $f$ must be stable. \qed

The next result is the tool needed to link stability and universality of maps of compacta into the Hilbert cube.

**THEOREM 2.** A map $f: X \to I^\infty$ from a compact metric space into the Hilbert cube is universal if and only if each composition $p_n \circ f: X \to I^n$ is universal.

**PROOF.** Assume that $f$ is universal. Fix a positive integer $n$. Consider the Hilbert cube as $I^n \times \prod_{j>n} [-1, 1]_j$, and let a map $g: X \to I^n$ be given. By choosing a point $y_j$ in $[-1, 1]_j$ for each $j > n$, we may assume that $g$ is a map into the Hilbert cube. Thus, since $f$ is universal, there exist a point $p$ in $X$ so that $g(p) \times \prod_{j>n} \{y_j\} = f(p)$. Thus, $g(p) = p_n \circ f(p)$, and $p_n \circ f$ is shown to be universal.

For the converse, assume that the Hilbert cube has the metric given by $d(y, y') = \sum_{i=1}^{\infty}(|y_i - y'_i|/2^i)$. Suppose that $p_n \circ f$ is universal for each positive integer $n$. If $f$ were not universal, then there exists a map $g: X \to I^\infty$ so that for every point $p$ in $X$, $g(p) \neq f(p)$. By the compactness of $X$, there exists a number $\delta > 0$ so that for every point $p$ in $X$ the distance $d(g(p), f(p)) > \delta$. But we can choose a positive integer $N$ so that the diam$(\prod_{i=N+1}^{\infty} [-1, 1]_i) < \delta$. This would imply that, for any point $p$ in $X$, $p_N \circ g(p) \neq p_N \circ f(p)$ contradicting the universality of $p_N \circ f(p)$. Therefore, $f$ must be universal. \qed

We are now ready to obtain the results mentioned at the end of the previous section.

**COROLLARY 1.** A map $f: X \to I^\infty$ from a compact metric space into the Hilbert cube is stable if and only if it is universal.

**PROOF.** This is immediate from Theorems 1 and 2. \qed
COROLLARY 2. If \( f : X \to I^\infty \) is stable and \( h \) is a self-homeomorphism of \( I^\infty \), then \( h \circ f \) is also stable.

PROOF. Clearly, universality of maps is preserved by composition with self-homeomorphisms of \( I^\infty \). By Corollary 1, the same is true for stable maps. \( \square \)

COROLLARY 3. Let \( \{(C_i, D_i)\}_{i=1}^\infty \) be an essential family for a strongly infinite-dimensional compact space \( X \), and let \( f \) be a map from \( X \) into \( I^\infty \) so that \( C_i = f^{-1}(A_i) \) and \( D_i = f^{-1}(B_i) \) for each \( i \). If \( h \) is any self-homeomorphism of \( I^\infty \), then \( \{(h \circ f)^{-1}(A_i), (h \circ f)^{-1}(B_i)\}_{i=1}^\infty \) is also an essential family for \( X \).

PROOF. This follows immediately from Corollary 2 and Walsh’s characterization of stability mentioned in §1. \( \square \)

REFERENCES


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