PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

G_κ SUBSPACES OF HYADIC SPACES

MURRAY G. BELL

(Communicated by Dennis Burke)

ABSTRACT. A hyadic space is a continuous image of a hyperspace of a compact space. For an infinite cardinal κ, an intersection of at most κ many open subsets of X is called a G_κ subset of X. We construct, in ZFC, a compact separable space of uncountable π-weight and of cardinality continuum. This space is a G_ω subset of a hyadic space. We show that a compact space that does not contain any convergent sequences and which contains the Stone-Čech compactification of the countable discrete space cannot be imbedded as a G_κ subset, where κ is less than the continuum, of any hyadic space.

1. Introduction. All spaces considered are assumed to be Hausdorff. We will always use the word *small* to mean cardinality at most continuum. The continuum is denoted by c, 0-dim is an abbreviation for zero-dimensional, $A\triangle B$ denotes the symmetric difference $(A-B)\cup(B-A)$ and $\mathscr{P}(X)$ denotes the set of all subsets of X. The Stone-Čech compactification of X will be denoted by βX. $\beta\omega$ is this compactification of the countable discrete space ω and ω^* is the subspace consisting of all nonisolated points. An intersection of κ many open sets will be called a G_κ set. A compact semilattice is a compact space that has a continuous binary operation that is idempotent, commutative and associative. If P is a partial order under $\leq$ and $x \in P$, then we put $\uparrow x = \{y \in P \colon x \leq y\}$ and $\downarrow x = \{y \in P \colon y \leq x\}$. The cofinality of a poset P is the least cardinality of a subset D such that for each $p \in P$, there exists $d \in D$ with $p \leq d$.

In §2, we deal with some basics of hyadic spaces. In §3, we construct a small compact space that is separable and has uncountable π-weight. This space is a G_ω subset of a hyadic space. In §4, we show that a space like $\beta\omega$ cannot be embedded as a G_κ subset, where $\kappa < c$, of any hyadic space. §5 lists some open problems not mentioned in earlier sections.

2. Hyadic spaces. Hyadic spaces were introduced by van Douwen [**3**] where he showed that every nontrivial G_ω subset of a hyadic space contained the limit point of some nontrivial convergent sequence. This result has been extended by Bell and Pelant [**2**] to: Every nonisolated point in a hyadic space is the endpoint of some infinite cardinal subspace. Let Y be a compact space and let $H(Y)$ denote the hyperspace of all nonempty closed subsets of Y with the Vietoris topology. The Vietoris topology is compact and the union map $U\colon H(Y)\times H(Y)\to H(Y)$ is continuous. Thus, $H(Y)$ is a compact semilattice. A continuous image of a $H(Y)$

Received by the editors January 27, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54D30; Secondary 54A25.

Key words and phrases. Hyadic, G_κ, semilattice, separable, cardinality, π-weight.

This research was supported by Grant No. U0070 from the Natural Sciences and Engineering Research Council of Canada.

where Y is compact is called a hyadic space. The hyadic spaces include all spaces with a binary normal subbase van Mill **[6]** and all generalized dyadic spaces (defined in §3).

A rich source of 0-dim hyadic spaces are the following spaces of filters. Let S be a nonempty collection of sets with $\bigcup S \in S$. A filter F of S is a nonempty subset of S such that if R is a finite subset of F then $\bigcap R \neq \emptyset$ and if R is a finite subset of F, $s \in S$ and $\bigcap R \subseteq s$ then $s \in F$. Define $\mathrm{Fil}(S)$ to be $\{F\colon F \text{ is a filter of } S\}$. Endow $\mathrm{Fil}(S)$ with the topology that has $\bigcup\{\{s^+, s^-\}\colon s \in S\}$ as a closed subbase where $s^+ = \{F\colon s \in F\}$ and $s^- = \mathrm{Fil}(S) - s^+$. $\mathrm{Fil}(S)$ is compact (by Alexander's Lemma), 0-dim and Hausdorff. It is seen that if $\bigcup S \in T \subseteq S$ then the mapping $\psi(F) = F \cap T$ is a continuous map of $\mathrm{Fil}(S)$ onto $\mathrm{Fil}(T)$. This fact, together with the fact that $\mathrm{Fil}(\mathscr{P}(\kappa))$ is naturally homeomorphic to $H(\beta\kappa)$, implies that each $\mathrm{Fil}(S)$ is a hyadic space.

Indeed, the class of spaces $\mathrm{Fil}(S)$ is precisely the class of all compact 0-dim semilattices; Hofmann, Mislove, and Stralka **[5]**. Hence, we see that hyadic spaces are precisely the continuous images of compact 0-dim semilattices.

EXAMPLES. If S is a disjoint collection of sets together with $\bigcup S$ then $\mathrm{Fil}(S)$ is the Alexandroff one-point compactification of the discrete space of size $|S|$. If S is an independent family of sets together with $\bigcup S$, then $\mathrm{Fil}(S)$ is the Cantor cube of weight $|S|$. If S is a decreasing κ-chain where κ is a cardinal, then $\mathrm{Fil}(S)$ is the cardinal space $\kappa + 1$. Let T be a tree with a smallest element and such that whenever t is in a limit level of T then t is uniquely determined by all elements less than t. There are two hyadic spaces associated with T which have received a lot of attention. Put $T\downarrow = \{\downarrow t\colon t \in T\} \cup \{T\}$ and $T\uparrow = \{\uparrow t\colon t \in T\}$. Upon identification of the filters and the topology, $\mathrm{Fil}(T\downarrow)$ is seen to be the one-point compactification of the usual locally compact tree topology on the nodes of T. Whereas, $\mathrm{Fil}(T\uparrow)$ is seen to be the space of all paths of T endowed with the subspace topology from 2^T.

3. A small compact separable space of uncountable π-weight. A π-base for a space is a collection of nonempty open sets such that every nonempty open set of the space contains one from the collection. The π-weight of a space is the least cardinal of some π-base. A space is G_ω scattered if it is scattered in its G_ω topology, i.e., every subspace has a G_ω point.

Much work has been done concerning the distinction between ccc spaces and separable spaces. In this section, we consider one aspect of the distinction between separable spaces and spaces of countable π-weight. The simplest example of a compact separable space of uncountable π-weight is 2^κ where κ is small uncountable. 2^κ has π-weight κ. However, if we now ask for such a space with small cardinality, we have an interesting problem. It was this question which initiated this research. Of course, if 2^{ω_1} is small then there is no problem; but what if it is not small? What we really want is an example which does not even map onto I^{ω_1}. Thus, by a result of Sapirovskii **[7]**, we do not want a closed subspace consisting entirely of points of uncountable π-character. We will construct, in ZFC, a compact separable space of uncountable π-weight which is G_ω scattered. Since for scattered spaces, cardinality is at most weight and the G_ω topology of any space of small weight has small weight, a compact separable space which is G_ω scattered necessarily must have small cardinality.

A useful construction of compact 0-dim Hausdorff space is: Let S be a collection of sets and let p be a property modifying subsets T of S. Further, assume that a subset T of S has p iff every finite subset of T has p. Then $X(S,p) = \{T \subseteq S : T \text{ has } p\}$, viewed as a subspace of 2^S, via characteristic functions, is closed in 2^S, hence is compact. In Bell [1] we investigated continuous images of such spaces, which we called generalized dyadic spaces. Spaces of this form have provided answers to many questions, notably in the ccc nonseparable realm. Unfortunately, no such space can help us here.

PROPOSITION 3.1. *Every separable $X(S,p)$ is either metric or else contains a copy of 2^{ω_1}.*

PROOF. Let D be a countable dense subset of $X(S,p)$. If there exists an uncountable $T \subseteq S$ of cardinality ω_1 and having p then $\mathscr{P}(T)$ is a copy of 2^{ω_1} contained in $X(S,p)$. If not, then every subset T of S having p is countable. $E = \{T \subseteq \bigcup D \colon T \text{ has } p\}$ is a closed subset of $X(S,p)$ containing D, hence $X(S,p) = E$ and thus is metric since $\bigcup D$ is a countable set. □

Now, we prove some general facts about certain G_ω subsets of a particular hyadic space. Put $P = \bigcup\{2^A \colon A \subseteq \omega\}$ and put $N = \{p \in P\colon \operatorname{dom} p \text{ is finite}\}$. A subset R of P is called compatible if $\bigcup R \in P$. A subset Q of P is called closed if $\bigcup F \in Q$ for every finite compatible subset F of Q. A nonempty subset I of a closed Q is called an ideal of Q if

(i) $\bigcup F \in I$ for every finite $F \subseteq I$, and

(ii) $p \in Q$, $q \in I$ and $p \subseteq q$ implies $p \in I$.

Note that from (i) it follows that I is compatible. Put $I(Q) = \{I\colon I \text{ is an ideal of } Q\}$.

For a closed Q, give 2^Q the Tychonoff product topology. Identifying each $I \in I(Q)$ with its characteristic function in 2^Q, we thus endow $I(Q)$ with the subspace topology from 2^Q. $I(Q)$ is closed in 2^Q and the intersection map $\cap$ from $I(Q)\times I(Q)$ to $I(Q)$ is continuous. Thus, $I(Q)$ is a compact 0-dim semilattice and therefore hyadic.

The parameter for our class of spaces will be a closed Q such that $N \subseteq Q \subseteq P$. For such a Q, put $J(Q) = \{I \in I(Q)\colon \bigcup I \text{ is a map from } \omega \text{ to } 2\}$. So, $J(Q)$ is a closed G_ω subspace of the compact semilattice $I(Q)$. Note that for K, $L \in J(Q)$, $K \cap L$ need not be in $J(Q)$ and it is unknown to the author whether, in general, $J(Q)$ must be hyadic. Indeed, whether hyadicity is closed G_ω hereditary is an open problem.

We get the following internal description of the topology of $J(Q)$: $\{F^+ \cap G^-: F$ and G are finite subsets of $Q\}$ is a base for the open sets, consisting of clopen sets, where for each $R \subseteq Q$ we define $R^+ = \{I \in J(Q)\colon R \subseteq I\}$ and $R^- = \{I \in J(Q)\colon I \cap R = \varnothing\}$. If $R = \{r\}$, then we write r^+ instead of $\{r\}^+$. It is readily seen, that the mapping $\lambda\colon J(Q) \to 2^\omega$ defined by $\lambda(I) = \bigcup I$ is a continuous onto map.

A key observation is that $\{q^+ : q \in Q\}$ is actually a π-base for $J(Q)$.

PROPOSITION 3.2. *$\{q^+ : q \in Q\}$ is a π-base for $J(Q)$.*

PROOF. Let $W = F^+ \cap G^- \neq \varnothing$, where F and G are finite subsets of Q. Pick $I \in W$ and let $f \in 2^\omega$ be such that $\{f|n\colon n \in \omega\} \subseteq I$. Put $A = \{q \in G\colon q \not\subseteq f\}$ and put $B = \{q \in G\colon q \subseteq f\}$. For each $q \in A$ choose $n_q \in \operatorname{dom} q$ with $q(n_q) \neq f(n_q)$.

Put $R = \bigcup\{\operatorname{dom} q\colon q \in F\} \cup \{n_q\colon q \in A\}$ and put $r = f|R$. Since $G \cap I = \varnothing$, for every $q \in B$ and for every finite subset H of ω we have that $\operatorname{dom} q \not\subseteq R \cup H$. This implies that we can choose, for each $q \in B$, an $m_q \in \operatorname{dom} q - R$ such that distinct q's yield distinct m_q's. Let s have domain $\{m_q\colon q \in B\}$ and satisfy that for each $q \in B$, $s(m_q) \neq q(m_q)$. Then, $\varnothing \neq (r \cup s)^+ \subseteq W$. □

COROLLARY 3.3. *The π-weight of $J(Q)$ equals the cofinality of Q under $\subseteq$.*

PROOF. This follows from the proposition since $q^+ \subseteq r^+$ iff $r \subseteq q$. □

COROLLARY 3.4. *The density of $J(Q)$ equals the least cardinal of an $D \subseteq J(Q)$ such that $\bigcup D = Q$.*

PROOF. This follows from the proposition since D is dense in $J(Q)$ iff $\bigcup D = Q$. □

EXAMPLE 3.5. Look at the largest space $J(P)$ and the associated $\lambda\colon J(P) \to 2^\omega$. The maximal ideals of P form a dense set of isolated points, one for each $f \in 2^\omega$, thus, $J(P)$ is a compactification of the discrete space of cardinality continuum. Upon an identification of $\lambda^{-1}(f)$, for an $f \in 2^\omega$, the reader will see that $\lambda^{-1}(f)$, as a subspace of $J(P)$ is homeomorphic to the disjoint union of $H(\omega^*)$ and one isolated point.

EXAMPLE 3.6. Let κ be an uncountable regular cardinal and let $\{A_\alpha\colon \alpha$ is a nonlimit ordinal $< \kappa\} \subseteq \mathscr{P}(\omega)$ be such that $\alpha < \gamma$ implies that $A_\alpha - A_\gamma$ is finite and that $A_\gamma - A_\alpha$ is infinite. Assume that $A_0 = \varnothing$. For each $A \subseteq \omega$, define $\pi(A) = \sup\{\alpha < \kappa\colon A \cap A_\alpha$ is finite$\}$. Define $Q = \{p \in P\colon$ there exists $\alpha < \kappa$ with $A_\alpha \triangle \operatorname{dom} p$ and $p^{-1}(1)$ both finite sets$\}$. Q is a closed subset of P containing N. For each $q \in Q$, define $\delta(q) =$ that $\alpha < \kappa$ such that $A_\alpha \triangle \operatorname{dom} q$ is finite.

CLAIM 1. *$J(Q)$ is separable.*

PROOF. For each finite subset F of ω put f_F equal to that element of 2^ω with $f_F^{-1}(1) = F$ and put $I_F = \{q \in Q\colon q \subseteq f_F\}$. $I_F \in J(Q)$ and $Q = \bigcup\{I_F\colon F$ is a finite subset of $\omega\}$. By Corollary 3.4, $J(Q)$ is separable. □

CLAIM 2. *The π-weight of $J(Q)$ is κ.*

PROOF. By Corollary 3.3, we must show that the cofinality of Q is κ. Since $|Q| = \kappa$, we must show that the cofinality of Q is not $< \kappa$. But if $R \subseteq Q$ has cardinality $< \kappa$, since κ is regular, there exists $\gamma < \kappa$ such that for each $q \in R$, $\delta(q) < \gamma$. Let $p\colon A_\gamma \to 2$ be the constant function 0. Then $p \in Q$ and $p \not\subseteq q$ for any $q \in R$. □

CLAIM 3. $J(Q)$ is G_ω scattered.

PROOF. It suffices to show that $J(Q)$ is covered by scattered G_ω subsets. For each $f \in 2^\omega$, $\lambda^{-1}(f)$ is a G_ω subset of $J(Q)$ and $J(Q) = \bigcup\{\lambda^{-1}(f)\colon f \in 2^\omega\}$. We claim that $\lambda^{-1}(f)$ is homeomorphic to the compact ordinal space $[0, \pi(f^{-1}(1))]$, which, of course, is scattered. The homeomorphism is given by the following h: $h(I) = \sup\{\alpha < \kappa\colon$ there exists $q \in I$ with $\delta(q) = \alpha\}$ where $\bigcup I = f$. It was to get this map one-to-one that we took nonlimit ordinals at the start. □

COMMENT. Our technique enables us to get a compact separable G_ω scattered space of π-weight κ, for each regular κ such that there exists a κ-chain in the Boolean algebra $\mathscr{P}(\omega)$ modulo finite sets. In our context G_ω scattered was important since it implied small cardinality. We have two open questions. Does it follow from ZFC that there exists a compact separable space of small cardinality and of maximum

π-weight c? Is it consistent with ZFC that there exists a compact separable G_ω scattered space of singular π-weight?

4. The character of $\beta\omega$ in a hyadic space. For each $A \in H(Y)$ and for each $\mathscr{A} \subseteq H(Y)$, put

$$\langle A, \mathscr{A}\rangle = \{B \in H(Y) \colon B \subseteq A \text{ and for each } C \in \mathscr{A}, B \cap C \neq \varnothing\}.$$

Then $\mathscr{S} = \{\text{all such } \langle A, \mathscr{A}\rangle\}$ is a closed subbase of $H(Y)$, which is closed under arbitrary intersections. We call $\mathscr{S}$ the canonical subbase. The map defined by $\pi(\mathscr{C}) = \bigcup \mathscr{C}$ is a continuous map of $H(H(Y))$ onto $H(Y)$. This is a special case of the similarly proven fact that if $S \in \mathscr{S}$ then $\pi(\mathscr{C}) = \bigcup \mathscr{C}$ defines a continuous map of $H(S)$ onto S. In particular, each member of $\mathscr{S}$ is hyadic.

DEFINITION. A κ-sieve of a space X is a cover $\mathscr{C}$ of at most κ many closed sets such that for each $x \in X$, $\bigcap\{C \in \mathscr{C} \colon x \in C\}$ has cardinality at most κ.

LEMMA 4.1. *If X is a compact space with a κ-sieve, then every closed subspace F of X contains a relative G_κ point.*

PROOF. Let $\mathscr{C}$ be a κ-sieve of X. For each $p \in F$ and for each $C \in \mathscr{C}$ such that $p \in X - C$ choose an open set $U_p(C)$ of X such that $p \in U_p(C)$ and $U_p(C) \cap C = \varnothing$. For each $p \in F$ choose a closed G_κ subset G_p of F such that $p \in G_p$ and $G_p \subseteq \bigcap\{F \cap U_p(C) \colon C \in \mathscr{C} \text{ and } p \in X - C\}$.

If $q \in F$, $A \subseteq F$ and $q \in \bigcap\{G_p \colon p \in A\}$ then $|A| \leq \kappa$. This is so because $A \subseteq \bigcap\{C \in \mathscr{C} \colon q \in C\}$. Let M be a subset of F maximal with respect to the property that $\{G_p \colon p \in M\}$ has the finite intersection property. By compactness, $\bigcap\{G_p \colon p \in M\} \neq \varnothing$ and hence $|M| \leq \kappa$. By maximality of M, $\bigcap\{G_p \colon p \in M\} \subseteq M$. Hence every point in $\bigcap\{G_p \colon p \in M\}$ is a G_κ point of F. □

THEOREM 4.2. *If X is a compact space which contains $\beta\omega$ and which contains no convergent sequences, then X cannot be embedded as a G_κ subset, where $\kappa < c$, of a hyadic space. Neither can βX if X is not pseudocompact.*

PROOF. Assume that Y is compact, $\mathscr{S}$ is the canonical subbase of $H(Y)$, ψ is a continuous map of $H(Y)$ onto K and that X is a G_κ subset of K. Then $\psi^{-1}[X]$ is a closed G_κ subset of $H(Y)$. Consequently, $\psi^{-1}[X] = \bigcap\{\bigcup \mathscr{R}_\alpha \colon \alpha < \kappa\}$ where the $\mathscr{R}_\alpha$'s are finite subsets of $\mathscr{S}$.

If there exists an $f \in \Pi\{\mathscr{R}_\alpha \colon \alpha < \kappa\}$ such that $\psi[\bigcap\{f(\alpha) \colon \alpha < \kappa\}]$ is infinite then $\psi[\bigcap\{f(\alpha) \colon \alpha < \kappa\}]$ is an infinite hyadic subspace of X and therefore X contains a convergent sequence by van Douwen's theorem.

Otherwise, for each $f \in \Pi\{\mathscr{R}_\alpha \colon \alpha < \kappa\}$ we have $\psi[\bigcap\{f(\alpha) \colon \alpha < \kappa\}]$ finite. Put $\mathscr{C} = \{\psi[\bigcap\{s(\alpha) \colon \alpha \in R\}] \colon s \in \Pi\{\mathscr{R}_\alpha \colon \alpha \in R\}$ and R is a finite subset of $\kappa\}$. If $x \in X$ and $\psi(y) = x$ then there exists $f \in \Pi\{\mathscr{R}_\alpha \colon \alpha < \kappa\}$ such that $y \in \bigcap\{f(\alpha) \colon \alpha < \kappa\}$. Therefore, $x \in \psi[\bigcap\{f(\alpha) \colon \alpha < \kappa\}]$. Since $H(Y)$ is compact $\psi[\bigcap\{f(\alpha) \colon \alpha < \kappa\}] = \bigcap\{\psi[\bigcap\{f(\alpha) \colon \alpha \in R\}] \colon R \text{ is a finite subset of } \kappa\}$. Hence for every $x \in X$, $\bigcap\{C \in \mathscr{C} \colon x \in C\}$ is finite and because $|\mathscr{C}| \leq \kappa$ we deduce that $\{X \cap C \colon C \in \mathscr{C}\}$ is a κ-sieve of X. By the previous lemma, every closed subspace of X contains a relative G_κ point. Therefore, X cannot contain $\beta\omega$ since $\beta\omega$ contains the absolute of 2^c, Efimov [**4**] and no point of the absolute of 2^c is a G_κ point, if $\kappa < c$.

Finally, if X is not pseudocompact then βX maps onto βR, where R is the space of real numbers. If βX was a G_κ subset of a hyadic space then βR would be one also, via the adjunction space. Hence $\beta R - R$ would be a G_κ subset of a hyadic space. But $\beta R - R$ contains $\beta\omega$ and does not contain a convergent sequence. Contradiction. □

5. Problems. 1. Is every compact Hausdorff semilattice the continuous image of a compact Hausdorff 0-dim semilattice?

2. If a separable hyadic space has uncountable π-weight must it contain a copy of 2^{ω_1}?

3. If a hyadic space does not contain the ordinal space ω_1 must it be a Frechet-Urysohn space (taken from [**2**])?

4. To find, in ZFC, a first countable or a weight ω_1 compact space which is not hyadic.

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Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2