CONCERNING PERIODIC POINTS
IN MAPPINGS OF CONTINUA

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ABSTRACT. In this paper we present some conditions which are sufficient for a mapping to have periodic points.

THEOREM. If \( f \) is a mapping of the space \( X \) into \( X \) and there exist subcontinua \( H \) and \( K \) of \( X \) such that (1) every subcontinuum of \( K \) has the fixed point property, (2) \( f[K] \) and every subcontinuum of \( f[H] \) are in class \( W \), (3) \( f[K] \) contains \( H \), (4) \( f[H] \) contains \( H \cup K \), and (5) if \( n \) is a positive integer such that \( (f|H)^{-n}(K) \) intersects \( K \), then \( n = 2 \), then \( K \) contains periodic points of \( f \) of every period greater than 1.

Also included is a fixed point lemma:

LEMMA. Suppose \( f \) is a mapping of the space \( X \) into \( X \) and \( K \) is a subcontinuum of \( X \) such that \( f[K] \) contains \( K \). If (1) every subcontinuum of \( K \) has the fixed point property, and (2) every subcontinuum of \( f[K] \) is in class \( W \), then there is a point \( x \) of \( K \) such that \( f(x) = x \).

Further we show that: If \( f \) is a mapping of \([0,1]\) into \([0,1]\) and \( f \) has a periodic point which is not a power of 2, then \( \lim\{[0,1],f\} \) contains an indecomposable continuum. Moreover, for each positive integer \( i \), there is a mapping of \([0,1]\) into \([0,1]\) with a periodic point of period \( 2^i \) and having a hereditarily decomposable inverse limit.

1. Introduction. In his book, An Introduction to Chaotic Dynamical Systems [3, Theorem 10.2, p. 62], Robert L. Devaney includes a proof of Sarkovskii's Theorem. Consider the following order on the natural numbers: \( 3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ \cdots \succ 2^2 \cdot 3 \succ 2^2 \succ 3 \succ 2^2 \cdot 5 \succ \cdots \succ 2^3 \cdot 3 \succ 2^3 \succ 5 \succ \cdots \succ 2^3 \succ 2^2 \succ 2 \succ 1 \). Suppose \( f: R \to R \) is continuous. If \( k \succ m \) and \( f \) has a periodic point of prime period \( k \), then \( f \) has a periodic point of period \( m \). In working through a proof of this theorem for \( k = 3 \), the author discovered the main result of this paper—Theorem 2. For an alternate proof of Sarkovskii's Theorem for \( k = 3 \), see also [7]. For a further look at this theorem for ordered spaces see [13].

By a continuum we mean a compact connected metric space and by a mapping we mean a continuous function. By a periodic point of period \( n \) for a mapping \( f \) of a continuum \( M \) into \( M \) is meant a point \( x \) such that \( f^n(x) = x \). The statement that \( x \) has prime period \( n \) means that \( n \) is the least integer \( k \) such that \( f^k(x) = x \). A continuum \( M \) is said to have the fixed point property provided if \( f \) is a mapping of \( M \) into \( M \) there is a point \( x \) such that \( f(x) = x \). A mapping \( f \) of a continuum \( X \) onto a continuum \( M \) is said to be weakly confluent provided for each subcontinuum \( K \) of...
M some component of \( f^{-1}(K) \) is thrown by \( f \) onto \( K \). A continuum is said to be in Class \( W \) provided every mapping of a continuum onto it is weakly confluent. The continuum \( T \) is a triod provided there is a subcontinuum \( K \) of \( T \) such that \( T - K \) has at least three components. A continuum is atriodic provided it does not contain a triod. A continuum \( M \) is unicoherent provided if \( M \) is the union of two subcontinua \( H \) and \( K \), then the common part of \( H \) and \( K \) is connected. A continuum is hereditarily unicoherent provided each of its subcontinua is unicoherent. If \( f \) is a mapping of a space \( X \) into \( X \), the inverse limit of the inverse limit sequence \( \{ X_i, f_i \} \) where, for each \( i \), \( X_i \) is \( X \) and \( f_i \) is \( f \) will be denoted \( \lim \{ X, f \} \). For the inverse sequence \( \{ X_i, f_i \} \), the inverse limit is the subset of the product of the sequence of spaces \( X_1, X_2, \ldots \) to which the point \( (x_1, x_2, \ldots) \) belongs if and only if \( f_i(x_{i+1}) = x_i \).

There has been considerable interest in periodic homeomorphisms of continua where a homeomorphism \( h \) is called periodic provided there is an integer \( n \) such that \( h^n \) is the identity. Wayne Lewis has shown [8] that for each \( n \) there is a chainable continuum with a periodic homeomorphism of period \( n \). A theorem of Michel Smith and Sam Young [14] should be compared with Theorem 3 of this paper. Smith and Young show that if a chainable continuum \( M \) has a periodic homeomorphism of period greater than 2, then \( M \) contains an indecomposable continuum. In this paper we consider the question of the existence of periodic points in mappings of continua.

2. A fixed point theorem. The problem of finding a periodic point of period \( n \) for a mapping \( f \) is, of course, the same as the problem of finding a fixed point for \( f^n \). Not surprisingly, we need a fixed point theorem as a lemma to the main theorem of this paper. The following theorem, which the author finds interesting in its own right, should be compared with an example of Sam Nadler [11] of a mapping with no fixed point of a disk to a containing disk. A corollary to Theorem 1 is the well-known corresponding result for mappings of intervals.

THEOREM 1. Suppose \( X \) is a space, \( f \) is a mapping of \( X \) into \( X \), and \( K \) is a subcontinuum of \( X \) such that \( f[K] \) contains \( K \). If (1) every subcontinuum of \( K \) has the fixed point property, and (2) every subcontinuum of \( f[K] \) is in Class \( W \), then there is a point \( x \) of \( K \) such that \( f(x) = x \).

PROOF. Since \( f[K] \) is in Class \( W \) and \( K \) is a subset of \( f[K] \), there is a subcontinuum \( K_1 \) of \( K \) such that \( f[K_1] = K \). Then \( f[K_1]: K_1 \to K \) is weakly confluent since every subcontinuum of \( f[K] \) is in Class \( W \); thus there is a subcontinuum \( K_2 \) of \( K_1 \) such that \( f[K_2] = K_1 \). Since \( K_1 \) is in Class \( W \), \( f[K_2]: K_2 \to K_1 \) is weakly confluent; therefore there is a subcontinuum \( K_3 \) of \( K_2 \) such that \( f[K_3] = K_2 \). Continuing this process there exists a monotonic decreasing sequence \( K_1, K_2, K_3, \ldots \) of subcontinua of \( K \) such that \( f[K_{i+1}] = K_i \) for \( i = 1, 2, 3, \ldots \). Let \( H \) denote the common part of all the terms of this sequence and note that \( f[H] = H \), since \( f[H] = f[\bigcap_{i>0} K_i] = \bigcap_{i>0} f[K_i] = \bigcap_{i>0} K_i = H \). Since \( f[H] \) throws \( H \) onto \( H \) and \( H \) has the fixed point property, there exists a point \( x \) of \( H \) (and therefore of \( K \)) such that \( f(x) = x \).

REMARK. Note that (1) and (2) of the hypothesis of Theorem 1 are met if \( f[K] \) is chainable ([12, Theorem 4, p. 236 and 4], respectively), while (2) is met if \( f[K] \) is.
atriodic and acyclic [1] and (1) is met by planar, tree-like continua such that each two points of a subcontinuum \( L \) lie in a weakly chainable subcontinuum of \( L \) [10].

3. Periodic points. In this section we prove the main result of the paper.

**Theorem 2.** If \( f \) is a mapping of the space \( X \) into \( X \) and there exist subcontinua \( H \) and \( K \) of \( X \) such that (1) every subcontinuum of \( K \) has the fixed point property, (2) \( f[K] \) and every subcontinuum of \( f[H] \) are in class \( W \), (3) \( f[K] \) contains \( H \), (4) \( f[H] \) contains \( H \cup K \), and (5) if \( n \) is a positive integer such that \( (f|H)^{-n}(K) \) intersects \( K \), then \( n = 2 \), then \( K \) contains periodic points of \( f \) of every period greater than 1.

**Proof.** Suppose \( n \geq 2 \). There is a sequence \( H_1, H_2, \ldots, H_{n-1} \) of subcontinua of \( H \) such that \( f[H_1] = K \) (note that \( f[H] \) is weakly confluent) and \( f[H_{i+1}] = H_i \) for \( i = 1, 2, \ldots, n - 2 \) (in case \( n > 2 \)). There is a subcontinuum \( K_n \) of \( K \) so that \( f[K_n] = H_{n-1} \). Thus, \( f^n[K_n] = K \) and so \( f^n[K_n] \) contains \( K_n \), so, by Theorem 1, there is a point \( x \) of \( K_n \) such that \( f^n(x) = x \). We must show that if \( j < n \) then \( f^j(x) \) is not \( x \). If \( j < n \) and \( f^j(x) = x \), then \( j = n - 2 \) and \( x \) is in \( H_2 \). Since \( f^n(x) = x \) and \( f^{n-2}(x) = x \), \( f^2(x) = x \). Since \( x \) is in \( (f|H)^{-2}(K) \), \( x \) is in \( (f|H)^{-4}(K) \) and in \( K \) contrary to (5) of the hypothesis. Therefore, \( x \) is periodic of prime period \( n \).

**Remark.** If \( f \) is a mapping of the continuum \( M \) into itself and \( f \) has a periodic point of period \( k \), then the mapping of \( \lim\{M, f\} \) induced by \( f \) has periodic points of period \( k \), e.g. \( (x, f^{k-1}(x), \ldots, f(x), x, \ldots) \). Thus, although Theorem 2 does not directly apply to homeomorphisms, it may be used to conclude the existence of homeomorphisms with periodic points.

**Corollary.** If \( M \) is a chainable continuum, \( f \) is a mapping of \( M \) into \( M \), and there are subcontinua \( H \) and \( K \) of \( M \) such that \( f[K] = H \), \( f[H] \) contains \( H \cup K \), and if \( (f|H)^{-n}(K) \) intersects \( K \) then \( n = 2 \) then \( f \) has periodic points of every period.
EXAMPLE. Let $f$ be the mapping of the simple triod $T$ to itself given in [5]. The mapping $f$ is represented in Figure 1 above. Letting $H = [0, A/2]$ and $K = [B/3, B/2]$ it follows from Theorem 2 that $f$ has periodic points of every period.

EXAMPLE. Let $f$ be the mapping of the simple triod $T$ to itself given in [2]. The mapping $f$ is represented in Figure 2 below. Letting $H = [0, 3B/8]$ and $K = [C/32, C/8]$, it follows from Theorem 2 that $f$ has periodic points of every period.

![Figure 2](image)

EXAMPLE. Let $f$ be the mapping of the unit circle $S^1$ to itself given by $f(z) = z^2$. Letting $H = \{e^{i\theta} | 0 \leq \theta \leq 3\pi/4\}$ and $K = \{e^{i\theta} | \pi \leq \theta \leq 3\pi/2\}$, it follows from Theorem 2 that $f$ has periodic points of every period. Similarly, if $f$ is a mapping of $S^1$ onto itself which is homotopic to $z^n$ for some $n > 1$, then $f$ has periodic points of every period.

![Figure 3](image)

COROLLARY. If $f$ is a mapping of an interval to itself with a periodic point of period 3, then $f$ has periodic points of every period.
**Proof.** To see this it is a matter of noting that the hypothesis of Theorem 2 is met. We indicate the proof for one of two cases and leave the second similar case to the reader. Suppose $a$, $b$ and $c$ are points of the interval with $a < b < c$ and $f(a) = b$, $f(b) = c$ and $f(c) = a$ [the other case is $f(a) = c$, $f(b) = a$ and $f(c) = b$].

If $f^{-1}(c)$ is nondegenerate, then there exist mutually exclusive intervals $H$ and $K$ lying in $[b,c]$ and $[a,b]$, respectively, so that $f[H]$ is $[a,c]$ and $f[K]$ is $[b,c]$ and Theorem 2 applies.

Suppose $f^{-1}(c) = \{b\}$. Choose $K$ lying in $[a,b]$ and $H$ lying in $[b,c]$ so that $f[K] = [b,c]$ and $f[H] = [a,c]$. For each $i$, denote by $H_i$ the set $(f_i[H])^{-1}(K)$. Note that $a$ is not in $H_i$ for $i = 1, 2, 3, \ldots$ so $c$ is not in $H_i$ for $i = 2, 3, 4, \ldots$ and thus $b$ is not in $H_i$ for $i = 3, 4, \ldots$. Further, $b$ is not in $H_1$ since $c$ is not in $K$. Thus, if $H_i$ intersects $K$, then $i = 2$. Consequently, the hypothesis of Theorem 2 is met.

**Remark.** Condition (5) of Theorem 2 seems a bit artificial. A more natural condition the author experimented with in its place is a requirement that $H$ and $K$ be mutually exclusive. In fact, in each of the examples, the $H$ and $K$ given are mutually exclusive. However, replacing condition (5) with this proved to be undesirable in that the Sarkovskii Theorem for $k = 3$ is not a corollary to Theorem 2 if the alternate condition is used. That condition (5) may not be replaced by the assumption that $H$ and $K$ are mutually exclusive can be seen by the following. For the function $f: [0,1] \rightarrow [0,1]$, which is piecewise linear and contains the points $(0,0), (1,1)$ and $(1,0)$, there do not exist mutually exclusive intervals $H$ and $K$ such that $f[H]$ contains $H \cup K$ and $f[K]$ contains $H$. To see this suppose $H$ and $K$ are such mutually exclusive intervals. By Theorem 2, $K$ contains a periodic point of $f$ of period 3. Note that $f^3$ has only four fixed points: 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1. $\frac{2}{3}$ is a fixed point for $f$, $K$ must contain one of 0, $\frac{1}{2}$, and 1. We complete the proof by showing that each of these possibilities leads to a contradiction.

(1) Suppose 0 is in $K$. Then 1 is in $H$ since $f^{-1}(0) = \{1\}$ and $f[H]$ contains $K$. But since $f^{-1}(1) = \{\frac{1}{2}\}$, $\frac{1}{2}$ is in both $H$ and $K$.

(2) Suppose 1 is in $K$. Since $f^{-1}(1) = \{\frac{1}{2}\}$, $\frac{1}{2}$ is in $H$. Since $f^{-1}(\frac{1}{2}) = \{0, \frac{3}{4}\}$ and $H$ and $K$ do not intersect 0 is in $H$ and $\frac{3}{4}$ is in $K$. But, $f^{-1}(0) = \{1\}$ so 1 is in $H$.

(3) Suppose $\frac{1}{2}$ is in $K$. As before, one of 0 and $\frac{3}{4}$ is in $H$. Since $f^{-1}(0) = \{1\}$, if 0 is in $H$ then 1 is in both $H$ and $K$. Thus $\frac{3}{4}$ is in $H$. Then $f^{-1}(\frac{3}{4})$ contains two points, $\frac{5}{8}$ and one less than $\frac{5}{8}$, so $P_1 = \frac{5}{8}$ is in $H$. Since $f^{-1}(P_1)$ contains two points, $\frac{1}{8}$ and one between $\frac{5}{8}$ and $\frac{3}{4}$, $\frac{1}{8}$ is in $K$. Thus, $f^{-1}(\frac{1}{8}) = \frac{15}{16}$ is in $H$. Since $f^{-1}(\frac{15}{16})$ contains two points, $\frac{17}{32}$ and one less than $\frac{17}{32}$, $P_2 = \frac{17}{32}$ is in $H$. Continuing this process, we get a sequence $P_1, P_2, \ldots$ of points of $H$ which converges to $\frac{1}{2}$. Thus $\frac{1}{2}$ is in $H$.

**4. Periodic points and indecomposability.** In this section we show that under certain conditions the existence of a periodic point of period three in a mapping of a continuum $M$ to itself implies that $\lim\{M, f\}$ contains an indecomposable continuum. Of course the result is not true in general since a rotation of $S^1$ by 120 degrees yields a homeomorphism of $S^1$ and a copy of $S^1$ for the inverse limit.

**Theorem 3.** Suppose $f$ is a mapping of the continuum $M$ into itself and $x$ is a point of $M$ which is a periodic point of $f$ of period three. If $M$ is atriodic and hereditarily unicoherent, then $\lim\{M, f\}$ contains an indecomposable continuum.
Moreover, the inverse limit is indecomposable if \( \text{cl}(\bigcup_{i>0} f^i[M_1]) = M \), where \( M_1 \) is the subcontinuum of \( M \) irreducible from \( x \) to \( f(x) \).

**PROOF.** Suppose \( x \) is a periodic point of \( f \) of period three. Denote by \( M_1, M_2 \) and \( M_3 \) subcontinua of \( M \) irreducible from \( x \) to \( f(x) \), \( f(x) \) to \( f^2(x) \) and \( f^2(x) \) to \( x \), respectively. Note that since \( M \) is hereditarily unicoherent, \( M_1 \cap (M_2 \cup M_3) = (M_1 \cap M_2) \cup (M_1 \cap M_3) \) is a continuum, so there is a point \( p \) common to all three continua.

The three continua \( M_1 \cap M_2, M_2 \cap M_3 \) and \( M_1 \cap M_3 \) all contain the point \( p \) so, since \( M \) is atriodic, one of them is a subset of the union of the other two [15]. Suppose \( M_1 \cap M_2 \) is a subset of \((M_2 \cap M_3) \cup (M_1 \cap M_3) = M_3 \cap (M_1 \cup M_2) = M_3 \). (The last equality follows since \( M_3 \cap (M_1 \cup M_2) \) is a subcontinuum of \( M_3 \) containing \( x \) and \( f^2(x) \) and \( M_3 \) is irreducible between \( x \) and \( f^2(x) \)). Then, \( M_1 \cup M_2 \) is a subset of \( M_3 \) for if not there is a point \( t \) of \( M_1 \cup M_2 \) such that \( t \) is not in \( M_3 \). Since \( M_1 \cap M_2 \) is a subset of \( M_3 \), \( t \) is in \( M_1 \) or in \( M_2 \) but not in \( M_1 \cap M_2 \). Suppose \( t \) is in \( M_1 - (M_1 \cap M_2) \). Since \( t \) is not in \( M_3 \), \( t \) is in \( M_1 - (M_1 \cap M_3) \) and thus \( t \) is in \( M_1 \cap (M_2 \cup M_3) \) or \( M_1 \cap (M_2 \cup M_3) \) is a continuum.

Note that \( f[M_1] \) is a continuum containing \( f(x) \) and \( f^2(x) \), so \( f[M_1] \cap M_2 \) is a subcontinuum of \( M_2 \) containing these two points. Since \( M_2 \) is irreducible from \( x \) to \( f(x) \), \( f[M_1] \cap M_2 = M_2 \). Therefore, \( M_2 \) is a subset of \( f[M_1] \). Similarly, \( f[M_2] \) contains \( M_3 \) and \( f[M_3] \) contains \( M_1 \). However, since \( M_3 \) contains \( M_1 \cup M_2, M_2 \) contains \( x, f(x) \) and \( f^2(x) \), so \( f[M_3] \) contains \( M_1 \cup M_2 \cup M_3 \). Thus, \( f^{n+2}[M_1] \) contains \( f^{n+1}[M_2] \) which contains \( f^n[M_3] \) which contains \( M_1 \cup M_2 \cup M_3 \) for \( n = 1, 2, 3, \ldots \) and so \( \text{cl}(\bigcup_{i>0} f^i[M_1]) = \text{cl}(\bigcup_{i>0} f^i[M_2]) = \text{cl}(\bigcup_{i>0} f^i[M_3]) \). Then, \( H = \text{cl}(\bigcup_{i>0} f^n[M_1]) \) is a continuum such that \( f[H]: H \to H \). Denote by \( K \) the inverse limit, \( \text{lim}(\{H, f[H]\}) \). We show that \( K \) is indecomposable by showing the conditions of [6, Theorem 2, p. 267] are satisfied. Suppose \( n \) is a positive integer and \( e \) is a positive number. There is a positive integer \( k \) such that if \( t \) is in \( H \) then \( d(t, f^k[M_2]) < e \). Suppose \( C \) is a subcontinuum of \( H \) containing two of the three points, \( x, f(x) \) and \( f^2(x) \). Then \( C \) contains one of \( M_1, M_2 \) and \( M_3 \). In any case \( f^2[C] \) contains \( M_3 \), and thus, if \( m = k + 2 \), \( d(t, f^m[C]) < e \) for each \( t \) in \( H \). By Kuykendall's Theorem, \( K \) is indecomposable.

**THEOREM 4.** If \( f \) is a mapping of \([0,1]\) to \([0,1]\) and \( f \) has a periodic point whose period is not a power of 2, then \( \text{lim}\{([0,1], f)\} \) contains an indecomposable continuum. Moreover, for each positive integer \( i \), there exists a mapping which has a periodic point of period \( 2^i \) and hereditarily decomposable inverse limit.\(^1\)

**PROOF.** Suppose \( f \) has a periodic point which has period \( n \) and \( n \) is not a power of 2. Then, \( n = 2^j(2k+1) \) for some \( j, k \geq 0 \), and \( f^{2^i} \) has a periodic point of period \( 2k+1 \). By the Sarkovskii Theorem, \( f^{2^i} \) has a periodic point of period 6, so

\(^1\)Added in proof: Theorem 4 first appeared, with a slightly different proof, as Theorem 1 of *Chaos, periodicity, and snakelike continua* by Marcy Barge and Joe Martin in a publication (MSRI 014-84) of the Mathematical Sciences Research Institute, Berkeley, California in January, 1984.
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\[ g = (f^{2^j})^2 \] has a periodic point of period 3. Since \( \lim \{[0, 1], f\} \) is homeomorphic to \( \lim \{[0, 1], g\} \), by Theorem 3 \( \lim \{[0, 1], f\} \) contains an indecomposable continuum.

In the family of maps \( f_\mu(x) = \mu x(1-x) \), for \( 2 < \mu < \mu_c \approx 3.5699456 \ldots \) all the inverse limits for \( \mu \) in this range are hereditarily decomposable and for each power of 2, there is a map in this collection with a periodic point of period that power of 2. In fact for \( 2 < \mu < 3 \) the inverse limit is an arc, for \( 3 < \mu < \mu_c \) the inverse limit becomes, as \( \mu \) increases, first a sinusoid, then a sinusoid to a double sinusoid, etc. For more details on this, see [9].

REFERENCES


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