POINTWISE ESTIMATES FOR THE RELATIVE FUNDAMENTAL SOLUTION OF $\partial_b$

MICHAEL CHRIST

(Communicated by Walter Littman)

ABSTRACT. Consider a compact pseudoconvex CR manifold of dimension 3 and finite type, on which the operator $\partial_b$ has closed range in $L^2$. The relative fundamental solution of $\partial_b$ is the distribution-kernel for that operator which inverts $\partial_b$, modulo its kernel and cokernel. We derive pointwise bounds on this fundamental solution and its derivatives.

Let $M$ be a compact CR manifold of real dimension 3. We assume that $M$ is pseudoconvex and of finite type $m$, and that the $\partial_b$ operator on $M$ has closed range on $L^2$. The latter holds automatically when $M$ is the boundary of a smooth, bounded pseudoconvex domain in $C^2$. Fix a positive measure on $M$ with a smooth, nonvanishing density in local coordinates. Let $S$ denote the Szegő projection of $L^2$, with respect to this measure, onto the kernel $H_b$ of $\partial_b$ in $L^2$. $\partial_b$ maps (test) functions to sections of a bundle $B^{0,1}$; fix an inner product structure on the bundle and let $L^{2*}$ denote the Hilbert space of $L^2$ sections of $B^{0,1}$. Let $S^*$ denote the adjoint operator, let $S^*$ denote the orthogonal projection of $L^{2*}$ onto the kernel $H_{b^*} \subset L^{2*}$ of $\partial_{b^*}$, and let $K, K^*$ be the distribution-kernels for $S, S^*$ respectively. For definitions of all these terms and references see for instance [C], [FK], [K].

The hypothesis that $\partial_b$ has closed range means that $\text{Range}(\partial_b) = L^{2*} \cap \partial_b(L^2)$ is a closed subspace of $L^{2*}$, and that for each $f \in \text{Range}(\partial_b)$ there exists a unique $u \in L^2$ satisfying

$$\begin{cases} 
\partial_b u = f, \\
u_b \perp H_b.
\end{cases}$$

Moreover $\|u\|_2 \leq C\|f\|_2$. Therefore the operator $G$ which maps any $f \in L^{2*}$ to the unique $u \perp H_b$ satisfying $\partial_b u = (I - S^*)f$, is bounded from $L^{2*}$ to $L^2$. Let $L$ denote its distribution-kernel, the relative fundamental solution for $\partial_b$. The purpose of this article is to obtain certain pointwise bounds for $L$ and its derivatives. This is a continuation of the work [C] and is based on the results obtained there; we shall continue to employ the notation of that paper without full explanation. In particular the bounds we seek for $L$ are formulated in terms of a quasi-metric $\rho$ and a family of balls $B(x, r)$ on $M$, constructed and studied in the fundamental paper [NSW], which are induced by the CR structure in a natural way. In this connection $\bar{B}$ denotes the unit ball in $\mathbb{R}^3$, and for each $x \in M, r \in (0, C_M]$ there is given a special coordinate map $\phi_{x, r}$, a diffeomorphism of $\bar{B}$ onto $B(x, r)$.
denotes the measure of $B(x,r)$. For a summary of their relevant properties see section 15 of [C].

In local coordinates in $M$, $\bar{\partial}_b$ takes the form $X + iY$ where $X,Y$ are real, smooth vector fields, linearly independent at every point. Define, for $x,y$ in a coordinate patch, $\vartheta(x,y)$ to be the infimum of all $r$ such that there exists an absolutely continuous function $\psi$ from $[0,r]$ into the coordinate patch, with $\psi(0) = x$ and $\psi(r) = y$, such that for almost all $t$, $d\psi/dt = a(t)X(\psi(t)) + b(t)Y(\psi(t))$, with $a^2(t) + b^2(t) \leq 1$. Then $\vartheta(x,y)$ is finite, and there is a uniform inequality $C\rho(x,y) \leq \vartheta(x,y) \leq C'\rho(x,y)$. $B(x,r)$ is $\{y: \rho(x,y) < r\}$. Equivalent reformulations of the estimates below may be obtained by replacing $\rho$ by $\vartheta$ and $B(x,r)$ by $\{y: \vartheta(x,y) < r\}$; the measures of $B(x,r)$ and $\{y: \vartheta(x,y) < r\}$ are comparable, uniformly in $x$ and $r$.

We denote by $D$ any differential operator of the form $(X$ or $Y) \circ (X$ or $Y)\ldots$ and let $n$ be the number of factors of $(X$ or $Y)$, possibly zero. $D_x$ denotes such an operator acting in the $x$-variable, with $n$ factors, and $D_y$ acts in the $y$-variable and has $n'$ factors.

Our main result is

**Theorem 1.** $L$ is $C^\infty$ away from the diagonal

$$|D_x D_y L(x,y)| \leq C_{n,n'} r^{1-n-n'} \Lambda(x,r)^{-1}$$

uniformly for all $n,n'$ and $x \neq y \in M$, where $r = \rho(x,y)$.

An immediate consequence is

**Theorem 2.** Suppose that $f \in L^2$, $u \perp H_b$ and $\bar{\partial}_b u = f$. Suppose further that $f$ is bounded on some open set $U \subset M$. Then $u$ is Hölder continuous of order $m^{-1}$ on every compact subset of $U$.

This follows directly from the first theorem, by Theorem 14(b) of [RS]. Moreover Theorem 1 implies that $|u(x) - u(y)| \leq C \rho(x,y) \log(\rho(x,y)^{-1})$ as $\rho(x,y)$ tends to 0. (Recall that $\rho(x,y) \leq C|x - y|^\delta$, where $\delta = m^{-1}$ [NSW].) Under the hypothesis of type $m$, this is the best order of regularity that could be concluded, even if it were known that $Xu, Yu$ were separately bounded on $U$. Thus the results of Theorem 1 are fairly sharp. Theorem 2 has also been obtained by Fefferman and Kohn [FK].

To begin the proofs observe that $L$ is $C^\infty$ away from the diagonal. For the distribution-kernel for $I - S^*$ is $C^\infty$ off of the diagonal (see below), and the solution $u \perp H_b$ of $\bar{\partial}_b u = h$ is $C^\infty$ wherever $h$ is. See [K] (or [C]). It remains to examine $L$ near the diagonal.

Consider any distinct points $x_0, y_0$ in a common coordinate patch, close together. Let $c_1 \ll 1 \ll c_2$ be two constants depending only on $M$, very small and very large respectively. Let $r = \rho(x_0,y_0), B = B(x_0,c_2r), B_1 = B(y_0,c_1r), B_3 = B(y_0,2c_1r), B_2 = B(x_0,c_1r)$, and $B_4 = B(x_0,2c_1r)$. To analyze $L$ and its $x$-derivatives at $(x_0,y_0)$ we consider the map from $L^2(B_1)$ to $C^\infty(B_2)$ which sends any $f \in L^2$ supported on $B_1$ to $Gf$ restricted to $B_2$. Let $u = Gf$.

The first step is to analyze $(I - S^*)f$. In [C] was proved

**Theorem A.** $K^*$ is $C^\infty$ away from the diagonal and satisfies

$$|D_x D_y K^*(x,y)| \leq C_{n,n'} \rho(x,y)^{-n-n'} \Lambda(x,\rho(x,y))^{-1}$$

for all $D_x, D_y$. The same holds for $K$. 

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Let \( h = (I - S^*)f \) and \( \hat{h} = h \circ \phi_{x_0,c_2r} \) on \( \hat{B} \). Let \( \hat{X} \) and \( \hat{Y} \) be the pullbacks of \( X, Y \) and let \( \hat{D} = (\hat{X} \circ Y) \circ (\hat{X} \circ \hat{Y}) \ldots \) with \( n \) factors on \( \hat{B} \). From Theorem A and the restriction that \( f \) be supported on \( B_1 \) there easily follows

\[
\|Dh\|_{L^2(B_2)} \leq C_n r^{-n} \|f\|_2.
\]

Equivalently

**COROLLARY 3.**

\[
\|\hat{D}h\|_{L^2(\hat{B}_2)} \leq C_n A(x_0, r)^{-1/2} \|f\|_2
\]

for all \( n \geq 0 \).

For each \( x_0, y_0 \) there exists \( \hat{\psi} \in C_0^\infty(\hat{B}) \) with \( C^k \) norm bounded uniformly in \( x_0, y_0 \) for all \( k \), satisfying \( \hat{\psi} \equiv 1 \) on \( \phi^{-1}(B_4) \) and \( \hat{\psi} \equiv 0 \) on \( \phi^{-1}(B_3) \), where \( \phi = \phi_{x_0,c_2r} \). Let \( \hat{\psi} = \hat{v} \circ \phi^{-1} \), which may be viewed as a \( C^\infty \) function on \( M \) supported on \( B_4 \) by virtue of the compact support of \( \hat{\psi} \). Let \( v \in L^2^* \) be the unique solution of

\[
\begin{aligned}
(I\! - \! S)^* v = (I\! - \! S)(\hat{\psi} u), \\
v \perp H_{b^*}.
\end{aligned}
\]

\( I - S \) projects onto \( \text{Range}(\overline{\partial}_b^*) \), the orthocomplement of \( H_b \), so a solution exists.

**LEMMA 4.**

\[
\|\psi u\|_2 \leq C \|f\|_2 \\
\|v\|_{L^2(B)} \leq C r \|f\|_2
\]

uniformly for all \( x_0, y_0, f \).

This is an immediate consequence of

**PROPOSITION B [C].** If \( f \in \text{Range}(\overline{\partial}_b^*) \) and \( u \perp H_{b^*} \) satisfies \( \overline{\partial}_b u = f \) then

\[
\|u\|_{L^2(B(x,r))} \leq C r \|f\|_2
\]

uniformly for all \( x \in M, r > 0, f \). The corresponding estimate is valid for the \( \overline{\partial}_b^* \) equation.

\( v \) is introduced in order to obtain the factor of \( r^2 \) in Lemma 4, which permits the rescaling argument below. It would be more natural to consider the solution \( w \) of \( \overline{\partial}_b^* w = u \) with \( w \perp H_{b^*} \), but we do not know that \( \|w\|_{L^2(B)} \leq C r^2 \|f\|_2 \). Otherwise Theorem 1 would be an immediate consequence of the arguments in [C].

Restrict everything to \( B_4 \), and let \( z = (I - S)(\hat{\psi} u) \). Then

\[
\begin{aligned}
\overline{\partial}_b^* v = z, \\
\overline{\partial}_b z = h,
\end{aligned}
\]

with the \( L^2(B_4) \) norms satisfying, for all \( n, \)

\[
\|r^{-2} v\|_2 + \|r^{-1} z\|_2 + \|r^n Dh\|_2 \leq C_n \|f\|_{L^2(M)}.
\]

Now pull everything back to \( \hat{B} \) via \( \phi = \phi_{x_0, 2c_2r} \). Let \( \hat{g} = g \circ \phi \) for any \( g \) defined on \( B \), and let \( \hat{\partial} \) and \( \hat{\partial}_* \) be the pullbacks of \( \overline{\partial}_b \) and \( \overline{\partial}_b^* \), respectively. Then the equations rescale to

\[
\begin{aligned}
\hat{\partial}_*(r^{-2}\hat{g}) &= (r^{-1}\hat{z}), \\
\hat{\partial}(r^{-1}\hat{z}) &= \hat{h},
\end{aligned}
\]

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with the control

\[ \| r^{-2} \hat{v} \|_2 + \| r^{-1} \hat{z} \|_2 + \| \hat{D} h \|_2 \leq C_n \Lambda(x_0, r)^{-1/2} \| f \|_2 \]

for all \( n \), uniformly in \( x_0, y_0 \).

It is proved in [K] (see also [C]) that this implies

\[ \| r^{-1} \hat{z} \|_{C^k} \leq C_k \Lambda(x_0, r)^{-1/2} \| f \|_2 \]

on any fixed compact subset of \( \hat{B} \), for all \( k \). Therefore on the inverse image of \( B_2 \)

\[ \| \hat{D} r^{-1} \hat{z} \|_{L^\infty(B)} \leq C_n \Lambda(x_0, r)^{-1/2} \| f \|_2, \]

which is to say that

\[ \| D[(I - S)(\psi u)] \|_{L^\infty(B_2)} \leq C_n r^{1-n} \Lambda(x_0, r)^{-1/2} \| f \|_2 \]

for all \( D \).

\( D[u - (I - S)(\psi u)] \) may be estimated more directly, on \( B_2 \). Let \( u_j \) be the restriction of \( u \) to \( B(x_0, 2^j r) \setminus \bigcup_{i < j} B(x_0, 2^i r) \). \( u = (I - S)u \) since \( u \perp H_b \), so \( [u - (I - S)(\psi u)] = (I - S)( -\psi u) \). Hence on \( B_2 \)

\[ [u - (I - S)(\psi u)] = -\sum_{j=0}^\infty S u_j. \]

Fix any \( D_2 \) and let \( K_j \) be the restriction of \( D_2 K \) to \( \{(x, y) : x \in B_2 \text{ and } \rho(x, y) \sim 2^j r \} \) so that \( D_2 S u_j = \int K_j(x, y) u_j(y) \, dy \) on \( B_2 \). Then

\[ \| D_2 S u_j \|_{L^\infty(B_2)} \leq C \sup_x \| K_j(x, \cdot) \|_\infty \| u_j \|_1 \]

\[ \leq C(2^j r)^{-n} \Lambda(x_0, 2^j r)^{-1} \| u_j \|_2 \Lambda(x_0, 2^j r)^{1/2} \]

\[ \leq C(2^j r)^{1-n} \Lambda(x_0, 2^j r)^{-1/2} \| f \|_2. \]

We have used Theorem A to estimate \( K_j \) and Proposition B to estimate \( \| u_j \|_2 \), and have used the facts that \( \rho \) satisfies a quasi-triangle inequality, and that \( \Lambda(x, C2^j) \approx \Lambda(x_0, 2^j) \) for \( x \in B_2 \). \( \Lambda(x_0, 2^j r) \geq C 2^{4j} \Lambda(x_0, r) [C, \S 15] \), so

\[ \| D_2 [u - (I - S)(\psi u)] \|_{L^\infty(B_2)} \leq C r^{1-n} \Lambda(x_0, r)^{-1/2} \| f \|_2 \cdot \sum_{j \geq 0} 2^{j(1-n)} 2^{-2j} \]

\[ \leq C r^{1-n} \Lambda(x_0, r)^{-1/2} \| f \|_2. \]

Together with (3), since \( u(x) = \int L(x, y) f(y) \, dy \), this establishes

**Lemma 5.** For all distinct \( x_0, y_0 \in M \) and all \( D_2 \)

\[ \| D_2 L(x, \cdot) \|_{L^2(B_{(y_0, c_1 r}))} \leq C_n r^{1-n} \Lambda(x_0, r)^{-1/2} \]

for all \( x \in B(x_0, c_1 r) \) where \( r = \rho(x_0, y_0) \).
The next claim is that the same holds with the roles of the variables reversed:

\[ \|D_y L(\xi, y)\|_{L^2(B(x_0, c_1r))} \leq Cr^{1-n'}\Lambda(x_0, r)^{-1/2} \]

for all \( y \in B(y_0, c_1r) \). (\( \Lambda(x_0, r) \sim \Lambda(y_0, r) \) so the lack of symmetry is only apparent.)

Observe that the adjoint \( G^* \) of \( G \) is the operator which first maps any \( g \in L^2 \) to \( (I-S)g \), then sends it to the solution \( v \perp H_{b^*} \) of \( \overline{\partial}_b^* v = (I-S)g \); in other words the distribution-kernel for \( G^* \) is the relative fundamental solution for \( \overline{\partial}_b^* \). Since the whole machine applies equally well to \( \overline{\partial}_b \), as to \( \overline{\partial}_b^* \), (4) follows from a repetition of the proof of Lemma 5. To verify the observation note that \( G^* = (I-S^*)G^*(I-S) \) since \( G = (I-S)G(I-S^*) \). Thus it suffices to show that \( \overline{\partial}_b \circ G^* \) is the identity on the orthocomplement of \( H_{b^*} \). But \( G\overline{\partial}_b = (I-S) \) on test functions by definition, so \( \overline{\partial}_b \circ G^* = (I-S^*) \).

Finally pull \( L \) back to \( \hat{L} \) on \( \hat{B} \times \hat{B} \) via \( \phi_{x_0,c_1r} \times \phi_{y_0,c_1r} \). Lemma 5 and (4) pull back to

\[ \sup_{\xi} \|\hat{D}_\xi \hat{L}(\xi, \cdot)\|_2 \leq C_n r \Lambda(x_0, r)^{-1} \]

and

\[ \sup_{\eta} \|\hat{D}_\eta \hat{L}(\cdot, \eta)\|_2 \leq C_n r \Lambda(x_0, r)^{-1} \]

for all \( \hat{D}_\xi, \hat{D}_\eta \), where the suprema are taken over \( \hat{B} \). The additional factor of \( \Lambda(x_0, r)^{-1/2} \) comes from the change of variables. Thus

\[ \|\hat{L}\|_{C^n} \leq C_n r \Lambda(x_0, r)^{-1} \]

on any fixed compact subset of \( \hat{B} \times \hat{B} \). Passing back to \( M \times M \) gives

\[ |D_x D_y \hat{L}(x_0, y_0)| \leq C_{n,n'} r^{1-n-n'} \Lambda(x_0, r)^{-1} \]

for all \( D_x, D_y \) as desired.

Observe that because of the relations \( \overline{\partial}_b G = (I-S) \) and \( \overline{\partial}_b^* G^* = (I-S^*) \), Theorem 1 implies Theorem A. It also implies Proposition B.

I am grateful to D. Jerison and E. M. Stein for pointing out an error in the original manuscript.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024