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FINITE TYPE CONES SHAPED ON SPHERICAL SUBMANIFOLDS

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ABSTRACT. The only finite type cones shaped on spherical submanifolds are minimal ones.

1. Introduction. Submanifolds of finite type introduced in [**1**] can be considered as a generalization of minimal submanifolds of a sphere and minimal submanifolds of an Euclidean space. A submanifold M of an Euclidean space E^{m+1} is said to be of k-type if the position vector x of M in E^{m+1} can be expressed in the following form:

$$x = c + x_{i_1} + \cdots + x_{i_k}, \qquad \Delta x_{i_t} = \lambda_{i_t} x_{i_t},\ \lambda_{i_1} < \cdots < \lambda_{i_t}, \tag{1.1}$$

where c is a constant vector; $\lambda_{i_t} \in \mathbf{R}$ are constant and, Δ is the Laplacian of the submanifold M with respect to the induced metric. For a k-type submanifold M, there exists a polynomial $P(t)$ of degree k such that $P(\Delta)H = 0$, where H denotes the mean curvature vector of M in E^{m+1}. (See [**1**, **2**] for more details.)

For a compact submanifold M of the unit hypersphere S^m of radius 1 centered at the origin, J. Simons [**3**] proved that if M is minimal in S^m, then $CM - \{0\}$ is minimal in E^{m+1}, where CM denotes the cone over M. In terms of finite type terminology, Simons' result says that $CM - \{0\}$ is of 1-type when M is of 1-type.

In this article, we prove the following

THEOREM. *Let $x\colon M \to S^m$ be an isometric immersion of a compact Riemannian manifold M into S^m. Then the punctured cone $CM - \{0\}$ is of finite type if and only if M is minimal in S^m.*

2. Lemmas. Let M be a compact p-dimensional submanifold of S^m. The cone over M, CM, is defined by the following map: $M \times [0,1] \to E^{m+1}$; $(m,t) \to tm$. Let H and H' denote the mean curvature vectors of $CM - \{0\}$ in E^{m+1} and that of M in S^m, respectively, and denote by ∇ and ∇' the connections of E^{m+1} and M. For $m \in M$ we choose a local orthonormal frame $\{E_i\}_{i=1}^p$ such that $\nabla'_{E_i} E_j(p) = 0$. Let ξ be the unit vector field of CM given by $\partial/\partial t$. Then the integral curves of ξ are rays from the origin of E^{m+1}. By extending the fields $\{E_i\}_{i=1}^p$ along the rays via parallel translation, we obtain an orthonormal local frame field on CM, which we also denote by $\{E_i\}_{i=1}^p$.

If σ is the second fundamental form of M in S^m, then we have

$$\nabla_{E_i} E_i(m,t) = (1/t^2)\sigma(E_i,E_i)(m) - (1/t)\xi. \tag{2.1}$$

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So we have

LEMMA 1. *Let $x\colon M \to S^m$ be an isometric immersion of a closed Riemannian manifold M into S^m, and $CM - \{0\}$ the punctured cone over M. Then we have*

$$H(m,t) = (p/(p+1)t^2)H'(m). \tag{2.2}$$

PROOF. Because $\nabla_\xi \xi = 0$, (2.1) implies

$$H(m,t) = \left(\frac{1}{(p+1)}\right)\left\{\sum_{i=1}^{p} \nabla_{E_i} E_i + \nabla_\xi \xi\right\}^N = \left(\frac{1}{(p+1)t^2}\right)\{pH' - pt\xi\}^N,$$

where $(\ ,\ )^N$ denote the normal component. So one has (2.2). Q.E.D.

Now, we compute the Laplacian of the cone in terms of the Laplacian of M. We shall see that Lemma 6.1.2 of [**3**] remains valid without the minimality condition on M.

LEMMA 2. *Under the hypothesis of Lemma* 1, *for a smooth function f of $CM - \{0\}$ we have*

$$(\overline{\Delta} f)(m,t) = \left(\frac{1}{t^2}\right)\Delta f_t(m) - \left\{\frac{\partial^2 f}{\partial t^2}(m,t) + \frac{p}{t}\frac{\partial f}{\partial t}(m,t)\right\}, \tag{2.3}$$

where $\overline{\Delta}$ and Δ denote the Laplacians of $CM - \{0\}$ and M, respectively, and f_t is defined by $f_t(m,t) = f(m,t)$, $t \in (0,1]$.

PROOF. We choose $\{E_i\}_{i=1}^p$ and ξ as before. Then we get

$$E_i(f)(m,t) = (1/t)E_i(f_t)(m). \tag{2.4}$$

On the other hand, if $\nabla''_\xi \xi$ is the Riemannian connection of $CM - \{0\}$, then

$$\overline{\Delta} f(m,t) = -\left\{\xi\xi(f) - (\nabla''_\xi \xi)(f) + \sum_{i=1}^{p}[E_iE_i(f) - \nabla''_{E_i}E_i(f)]\right\}$$

and so by using (2.1) and (2.4) we have (2.3). Q.E.D.

Now, we put

$$a_n = 2n(p - 2n - 1), \qquad n = 1, 2, \ldots, \tag{2.5}$$

where p is the dimension of M. Let $S_J(x_1, \ldots, x_n)$ denote the elementary symmetric functions in $x_1, \ldots, x_n$, i.e.,

$$S_1(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i,\ S_2(x_1,\ldots,x_n) = \sum_{i<j} x_ix_j, \ldots, S_n(x_1,\ldots,x_n) = x_1\cdots x_n.$$

We also put $d_J^n = S_J(a_1, \ldots, a_n)$. Then we have

LEMMA 3. *If $x\colon M \to S^m$ is an isometric immersion of a compact Riemannian manifold M into S^m, then for any integer n we have*

$$\overline{\Delta}^n H = (p/(p+1)t^{2(n+1)})\{\Delta^n H' + d_1^n \Delta^{n-1} H' + d_2^n \Delta^{n-2} H' + \cdots + d_{n-1}^n \Delta H' + d_n^n H'\}. \tag{2.6}$$

PROOF. We put $H = (h_1, \dots, h_{m+1})$ and $H' = (h'_1, \dots, h'_{m+1})$. Then from (2.2) we have

$$h_i = (p/(p+1)t^2)h'_i, \qquad i = 1, \dots, m+1.$$

By using (2.3) one gets

$$\overline{\Delta}(h_i) = (p/(p+1)t^4)\{\Delta h'_i + (2p-6)h'_i\}, \qquad i = 1, \dots, m+1,$$

so that

$$\overline{\Delta} H = (p/(p+1)t^4)\{\Delta H' + (2p-6)H'\}.$$

We repeat the argument on the component functions of $\overline{\Delta} H$ to obtain

$$\overline{\Delta}^2 H = (p/(p+1)t^6)\{\Delta^2 H' + [4(p-5) + 2(p-3)]\Delta H' + 8(p-5)(p-3)H'\}.$$

Hence, (2.6) follows by induction. Q.E.D.

3. Proof of Theorem. Suppose there exists a minimal polynomial of degree k, $P(t) = t^k + c_1 t^{k-1} + \cdots + c_k$, such that

$$\overline{\Delta}^k H + c_1 \overline{\Delta}^{k-1} H + \cdots + c_k H = 0, \qquad c_i \in \mathbf{R}. \tag{3.1}$$

Then, by substituting (2.6) into (3.1), we have

$$\begin{aligned} \Delta^k H' &+ (d_1^k + c_1 t^2)\Delta^{k-1} H' + (d_2^k + c_1 d_1^{k-1} t^2 + c_2 t^4)\Delta^{k-2} H' \\ &+ \cdots + (d_k^k + c_1 d_{k-1}^{k-1} t^2 + \cdots + c_{k-1} d_1^1 t^{2k-2} + c_k t^{2k})H' = 0. \end{aligned} \tag{3.2}$$

Since equality (3.2) holds for every (m,t) in $CM - \{0\}$, (3.2) is true for any t in $(0,1]$ at every fixed point m in M. Therefore, we conclude that $H' = 0$. The converse follows from (2.2). Q.E.D.

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