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MORDELL-WEIL GROUPS OF GENERIC ABELIAN VARIETIES IN THE UNITARY CASE

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ABSTRACT. The generic fibre of a fibre system of polarized abelian varieties with level structure, and with endomorphism structure coming from a CM-field, is defined over the function field of the moduli space for the abelian varieties. We prove that the points on this generic abelian variety which are defined over the function field of the moduli space form a finite group. The methods of proof generalize those of *Mordell-Weil groups of generic abelian varieties*, Invent. Math. **81** (1985), 71–106, to which this paper is a sequel.

Introduction. In this paper we will consider fibre systems of polarized abelian varieties over $\mathbf{C}$ characterized by having a fixed CM-field embedded in their endomorphism algebras. If V is the moduli space for such abelian varieties (with level structure), and W is the fibre variety constructed in [**4**], then the fibre over the generic point of V is an abelian variety defined over the function field of V. We consider the points of this abelian variety which are defined over $\mathbf{C}(V)$. Our result is

THEOREM. *If* $\dim V > 0$, *then the Mordell-Weil group of the generic fibre is finite.*

Here, V is isomorphic to a product of domains of the form $\{r \times s$ complex matrices $Z | 1 - Z^t\overline{Z} > 0\}$ modulo the action of a discrete subgroup of a unitary group. The setting for which the theorem holds is described more precisely in §1.

The analogous theorem was proved for V a noncompact quotient of the complex upper half plane by Shioda (Theorem 5.1 of [**5**]). When V is a quotient of a Hilbert-Siegel space H_s^r and V is the moduli space for abelian varieties whose endomorphism algebras contain a fixed totally real field, CM-field, totally indefinite quaterion algebra over a totally real field, or quaternion algebra over a CM-field, the analogous theorem was proved in [**6**] (see also [**7**]). The present paper makes use of new methods to remove the restriction on the base variety V in the CM-field case.

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NOTATION. We will write $\mathrm{diag}(A_1, \ldots, A_s)$ for the matrix with blocks $A_1, \ldots, A_s$ on the diagonal, $I(r)$ for the identity matrix in $GL_r(\mathbf{Z})$, and ${}^t\gamma$ for the transpose of the matrix γ.

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1. Background. In this section we recall the set-up of [**2** and **4**]. Let K be a totally imaginary quadratic extension of a totally real number field F, let ρ be complex conjugation, and let $g = [F : \mathbf{Q}]$. We may view K as embedded in $\mathbf{C}^g \cong K \otimes_{\mathbf{Q}} \mathbf{R}$, and for $k = 1, \ldots, g$ we let τ_k be the projection of $K \otimes \mathbf{R}$ onto the kth factor. Fix an integer $m > 1$ and a matrix $T \in GL_m(K)$ with ${}^tT^\rho = -T$. Suppose $i(T^{\tau_k})$ has signature (r_k, s_k) for $k = 1, \ldots, g$. By renumbering the τ_k's, and replacing τ_k by $\rho\tau_k$ if necessary, we may assume $r_k \geq s_k$ for all k, and $r_1 = \min_k r_k$. We must assume T is indefinite, i.e., $r_1 \neq m$ (otherwise the base variety V is a single point and our theorem clearly fails). For $x = (x_1, \ldots, x_g) \in \mathbf{C}^g$, let $\Phi(x) = \operatorname{diag}(x_1 I(r_1), \bar{x}_1 I(s_1), \ldots, x_g I(r_g), \bar{x}_g I(s_g)) \in M_n(\mathbf{C})$, where $n = gm$. Fix a lattice $\mathscr{M} \subset K^m$ of rank $2n$ and so that $\operatorname{tr}_{K/Q}(\mathscr{M} T {}^t\mathscr{M}^\rho) \subset \mathbf{Z}$.

We can find $W \in GL_m(K)$ so that $WT{}^tW^\rho$ is diagonal. Therefore by adjusting the lattice $\mathscr{M}$, we may assume T is diagonal and of the form

$$T = \alpha \cdot \operatorname{diag}(a_1, \ldots, a_m) \text{ with } \alpha \in K^\times,\ \alpha^\rho = -\alpha, \text{ and } a_1, \ldots, a_m \in F^\times.$$

Let $J_k = \operatorname{diag}(I(r_k), -I(s_k))$, for $k = 1, \ldots, g$. There exist matrices $W_k \in GL_m(\mathbf{R})$ so that $iW_k T^{\tau_k} {}^tW_k = J_k$ for $k = 1, \ldots, g$. Further, we can choose W_k in the form $P_k \cdot D_k$ where D_k is diagonal and P_k is a permutation matrix defining a permutation of order two, π_k, on the set $\{1, \ldots, m\}$. Also we may rearrange $a_1, \ldots, a_m$ so that W_1 is diagonal—then $\pi_1 = 1$.

Let $\mathbf{H} = \mathbf{H}(T) = \bigoplus_{k=1}^g H(r_k, s_k)$ where

$$H(r, s) = \{r \times s \text{ matrices } Z \mid I(r) - Z{}^t\overline{Z} \text{ is positive hermitian}\}.$$

For $Z = (Z_1, \ldots, Z_g) \in \mathbf{H}$, let

$$X_k = \begin{pmatrix} I(r_k) & Z_k \\ {}^t\overline{Z}_k & I(s_k) \end{pmatrix} \quad W_k = \begin{pmatrix} u_{1,k} & \cdots & u_{m,k} \\ \bar{v}_{1,k} & \cdots & \bar{v}_{m,k} \end{pmatrix}$$

with $u_{j,k} \in \mathbf{C}^{r_k}, v_{j,k} \in \mathbf{C}^{s_k}$ for $k = 1, \ldots, g$ and let

$$R_l(Z, T) = {}^t({}^tu_{l,1}, {}^tv_{l,1}, \ldots, {}^tu_{l,g}, {}^tv_{l,g}) \quad \text{for } l = 1, \ldots, m.$$

For $x = (x_1, \ldots, x_m) \in (K \otimes \mathbf{R})^m$ let

$$\eta_T(x, Z) = \sum_{l=1}^m \Phi(x_l) R_l(Z, T).$$

Also fix elements $v_1, \ldots, v_s \in K^m$, let $\mathscr{O} = \{a \in K \mid a\mathscr{M} \subset \mathscr{M}\}$, and let $\mathscr{N} = \mathscr{M} + \sum_{j=1}^s \mathscr{O} v_j$. Let $G = \{\gamma \in GL_m(K \otimes \mathbf{R}) \mid {}^t\gamma^\rho T \gamma = T\}$, let $G_K = G \cap GL_m(K)$, and let $\Gamma = G_K \cap \{\gamma \in M_m(K) \mid M\gamma = \mathscr{M}, N(1 - \gamma) \subset \mathscr{M}\}$. As in [**6**] we may assume the v_j's have been chosen so that $\Gamma \subset SL_m(K)$ and Γ has no nonidentity elements of finite order. Let $G(r, s)$ be the unitary group $\{\gamma \in GL_m(\mathbf{C}) \mid {}^t\overline{\gamma} J \gamma = J\}$, where $J = \operatorname{diag}(I(r), -I(s))$. Then $G(r, s)$ acts on $\mathbf{H}(r, s)$: for $Z \in \mathbf{H}(r, s)$, and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(r, s)$$

with $a \in M_r(\mathbf{C})$ and $d \in M_s(\mathbf{C})$, we have $\gamma(Z) = (aZ + b)(cZ + d)^{-1}$. The map $\gamma \to ({}^tW_1^{-1} \gamma^{\tau_1} {}^tW_1, \ldots, {}^tW_g^{-1} \gamma^{\tau_g} {}^tW_g)$ is an isomorphism from G to $\bigoplus_{k=1}^g G(r_k, s_k)$, and defines an action of G on $\mathbf{H}$.

For $Z \in \mathbf{H}$, let $A_Z = C^n/\eta_T(\mathscr{M}, Z)$, let C_Z be the polarization of A_Z defined by T (see (11) of [**2**]), let $\theta_Z = \Phi$, and let $t_j(Z) = \eta_T(v_j, Z)$. Write $Q_Z = (A_Z, C_Z, \theta_Z, t_1(Z), \dots, t_s(Z))$. Then Q_Z is a polarized abelian variety of type $(K, \Phi, \rho, T, \mathscr{M}, v_1, \dots, v_s)$ (see Definition 1.2 of [**6**]). By 3.6 of [**4**] there is a map $\phi\colon \mathbf{H} \to \mathbf{P}^M$ inducing an embedding of $\mathbf{H}/\Gamma$ into a projective space $\mathbf{P}^M$ so that $V = \phi(\mathbf{H})$ is a nonsingular variety. Further, by 3.8 of [**4**], there is a fibre system $(V, W, \dots)$ of abelian varieties so that for $Z \in \mathbf{H}$ and $u = \phi(Z)$, the fibre $Q_u = (A_u, \dots)$ is isomorphic to Q_Z. The variety W is analytically isomorphic to $(\mathbf{H} \times \mathbf{C}^n)/(\Gamma \times \mathscr{M})$, where the group law and action are defined as in 3.14 of [**4**] or 1.4 of [**6**]. The fibre over a generic point of V is an abelian variety defined over the function field $\mathbf{C}(V)$ of V.

PROPOSITION 1. *The group of points over $\mathbf{C}(V)$ of the generic fibre is isomorphic to the group of holomorphic sections from V to W.*

PROOF. Corollary 2.2 of [**6**]. □

THEOREM I. *In the situation described above, the Mordell-Weil group over $\mathbf{C}(V)$ of the generic fibre is finite.*

When $m = 2$, the theorem was proved for $\dim V > 1$ in case II_c of the Main Theorem of [**6**] (see §5 of [**6**]), for $\dim V = 1$ and V compact in Case I_c of the Main Theorem of [**6**] (see §4.3 of [**6**]), and for $\dim V = 1$ and V not compact in Shioda's Theorem 5.1 of [**5**]. In what follows, we will assume $m \geq 3$.

2. Reduction to lower dimensions. Our method of proof is to pull back the fibre system to fibre systems of the same type, but with $m = 2$. We can then use the results for $m = 2$ to give a dense set of points in V whose images under sections are points of finite order, and whose fibres have CM by K^m. In §3 we show this suffices to prove Theorem I.

The method of proof here differs from §6 of [**6**] in one significant way: in §6.4 of [**6**], we pulled back the fibre system to only *one* fibre system over a lower dimensional base. That technique no longer suffices—it is now sometimes necessary to choose as many as $r_1 \cdot s_1$ different pullbacks, as can be seen by examining the case $iT = \mathrm{diag}(1, 1, -1), K = \mathbf{Q}(i)$.

Write 0 for the element of $\mathbf{H}$ with zero in every entry.

THEOREM 2. *Suppose $h\colon \mathbf{H} \to \mathbf{C}^n$ is a holomorphic map which induces a section from V to W. Then for every $\beta \in G_K$, $h(\beta(0))$ gives a point of finite order in $A_{\beta(0)}$, and for some θ_β and Φ', $(A_{\beta(0)}, C_{\beta(0)}, \theta_\beta)$ is of type (K^m, Φ').*

PROOF. Fix positive integers: $i \leq r_1; j \leq s_1$. Let $T_{ij} = \alpha \cdot \mathrm{diag}(a_i, a_{r_1+j})$. For $1 \leq k \leq g$ let (r'_k, s'_k) be the signature of $iT_{ij}^{\tau_k}$. Let $M_k = \pi_k(i)$, $N_k = \pi_k(r_1+j)$, and write $F_k(a, b)$ for the $r_k \times s_k$ matrix with one in position (a, b) and zeros elsewhere. Define a map ε $(= \varepsilon_{ij})$ from $\mathbf{H}(T_{ij}) = \bigoplus_{k=1}^{g} H(r'_k, s'_k)$ to $\mathbf{H}$ by $\varepsilon(z_1, \dots, z_g) = (Z_1, \dots, Z_g)$ with

$$Z_k = \begin{cases} z_k \cdot F_k(M_k, N_k - r_k) & \text{if } M_k \leq r_k \text{ and } N_k > r_k, \\ z_k \cdot F_k(N_k, M_k - r_k) & \text{if } M_k > r_k \text{ and } N_k \leq r_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note. Since $(r'_1, s'_1) = (1, 1)$, we have $\dim \mathbf{H}(T_{ij}) > 0$.

For $l = 1, \ldots, m$ define projection maps

$$p_l(x_1, \ldots, x_{mg}) = (x_l, x_{\pi_2(l)+m}, \ldots, x_{\pi_g(l)+(g-1)m}),$$

and let $p_{ij} = p_i \oplus p_{r_1+j}$. Observe that $p_{ij}(\eta_T(\mathcal{O}_K^m, \varepsilon(z))) = \eta_{T_{ij}}(\mathcal{O}_K^2, z)$. Also, let $T_l = \alpha a_l$ and observe that $p_l(\eta_T(\mathcal{O}_K^m, \varepsilon(z))) = \eta_{T_l}(\mathcal{O}_K, 0)$, for $l = 1, \ldots, m$ (here, $0 \in \mathbf{H}(T_l)$). Write $(A'_Z, C'_Z, \theta'_Z) = P'_Z$ for the abelian variety determined by $Z, T, \mathcal{O}_K^m$ (as in Theorem 1 of [**2**]). Then for $z \in \mathbf{H}(T_{ij})$,

$$A'_{\varepsilon(z)} \cong \mathbf{C}^n/\eta_T(\mathcal{O}_K^m, \varepsilon(z)) \cong \mathbf{C}^{2g}/\eta_{T_{ij}}(\mathcal{O}_K^2, z) \oplus \bigoplus_{\substack{l=1 \\ l \neq i, r_1+j}}^{m} \mathbf{C}^g/\eta_{T_l}(\mathcal{O}_K, 0).$$

Fix $\beta \in G_K$. By 2.8 of [**2**] (see also §6.3 of [**6**]), $(A_{\beta(Z)}, C_{\beta(Z)}, \theta_{\beta(Z)})$ is isogenous to P'_Z for $Z \in \mathbf{H}$. For $z \in \mathbf{H}(T_{ij})$, write λ_z for the isogeny from $A_{\beta\circ\varepsilon(z)}$ to $A'_{\varepsilon(z)}$, and Λ_z for the associated map from $\mathbf{C}^n$ to $\mathbf{C}^n$. As in §6 of [**6**], the map

$$h_{ij}(z) = p_{ij} \circ \Lambda_z \circ h \circ \beta \circ \varepsilon(z)$$

from $\mathbf{H}(T_{ij})$ to $\mathbf{C}^{2g}$ is holomorphic. We can construct a fibre system of polarized abelian varieties of type $(K, \Phi_{ij}, \rho, T_{ij}, \mathcal{O}_K^2$, representatives for $(1/M)\mathcal{O}_K^m/\mathcal{O}_K^m)$ with base $\mathbf{H}(T_{ij})/\Delta$ for Δ the principal congruence subgroup of level M in $\{\delta \in SL_2(\mathcal{O}_K) | {}^t\delta^\rho T_{ij} \delta = T_{ij}\}$, for a suitable level M so that Δ embeds in $\beta^{-1}\Gamma\beta$ in a natural way. Here, Φ_{ij} is defined by $\{(r'_k, s'_k)\}_{k=1}^g$. By Theorem 5.1 of [**5**] and Cases I_c and II_c of the Main Theorem of [**6**], h_{ij} is a section of finite order for this new fibre system. Thus, for every $z \in \mathbf{H}(T_{ij})$, $\beta \in G_K$, $1 \leq i \leq r_1$, $1 \leq j \leq s_1$, $h_{ij}(z)$ is a point of finite order in $\mathbf{C}^{2g}/\eta_{T_{ij}}(\mathcal{O}_K^2, z)$.

Now let $z = 0$ in $\mathbf{H}(T_{ij})$ (and then $\varepsilon(z) = 0$ in $\mathbf{H}$). Let $E_l = \mathbf{C}^g/\eta_{T_l}(\mathcal{O}_K, 0)$ for $1 \leq l \leq m$. E_l has CM by (K, Φ_l), where for $x \in K \otimes \mathbf{R}$, $\Phi_l(x) = \mathrm{diag}(x^{\delta_1}, \ldots, x^{\delta_g})$ with

$$\delta_k = \begin{cases} \tau_k & \text{if } \pi_k(l) \leq r_l, \\ \rho\tau_k & \text{if } \pi_k(l) > r_l. \end{cases}$$

Then $A'_0 \cong \bigoplus_{l=1}^m E_l$, and $\{\Phi_l\}_{l=1}^m$ defines a map $\Phi' \colon K^m \to M_n(\mathbf{C})$ and an embedding $\tilde{\theta}_0 \colon K^m \to \mathrm{End}(A'_0) \otimes \mathbf{Q}$. For $\beta \in G_K$, the map λ_0 defines an isogeny from $(A_{\beta(0)}, C_{\beta(0)}, \lambda_0^{-1}\tilde{\theta}_0\lambda_0)$ to $(A'_0, C'_0, \tilde{\theta}_0)$. For every i and j, $h_{ij}(0)$ gives a point of finite order in $E_i \times E_{r_1+j}$. Thus $\Lambda_o h(\beta(0))$ has finite order in A'_0. Consequently, for every $\beta \in G_K$, $h(\beta(0))$ has finite order in $A_{\beta(0)}$.

3. Conclusion of proof of Theorem I. By Theorem 5.3 of [**4**], the fibre system $(V, W, \ldots)$ is defined over a number field k_Ω so that for every $u \in V$, $k_\Omega(u)$ is the field of moduli of the fibre Q_u. For $\beta \in G_K$, let $\tilde{\beta} = \phi(\beta(0)) \in V$. Then $A_{\tilde{\beta}} \cong A_{\beta(0)}$.

PROPOSITION 3. *Suppose L is a subfield of $\mathbf{C}$ which is finitely generated over $\mathbf{Q}$ and contains k_Ω. There is a constant B_L so that for every $\beta \in G_K$, $|A_{\tilde{\beta}}(L(\tilde{\beta}))_{\mathrm{tors}}| \leq B_L$.*

PROOF. This follows immediately from Theorem 2 above, and Theorem A of §7.1 of [**6**] (see also Theorem 2 of [**7**]). □

PROPOSITION 4. *All sections are torsion.*

PROOF (SEE §7.8 OF [**6**]). Fix a section and choose L to be a field over which the section is defined. The result follows from Proposition 3 and the fact that $\{\tilde{\beta}|\beta \in G_K\}$ is dense in V. □

Theorem I now follows, either by a Mordell-Weil theorem for abelian varieties over function fields, or by bounding the number of holomorphic sections of finite order. Such a bound is given by Corollary 2.7 of [**6**], which depends on the following lemma (Lemma 2.4 of [**6**] when the base variety was a quotient of a Hilbert-Siegel space).

LEMMA 5. *There is an element* $\gamma \in \Gamma$ *so that* $\det(1-\gamma) \neq 0$.

We first take a more general setting. Suppose now that K is any number field. Write $K_{\mathbf{A}}$ for the finite adèles of K, and write $\mathscr{O}_v$ for the ring of integers of the completion K_v. Suppose $H \subset GL_m(\mathbf{C})$ is a linear algebraic group defined over K. We have the usual topological groups $H_K, H_{\mathbf{A}}$ $(= H_{K_{\mathbf{A}}}), H_v$ $(= H_{K_v}), H(\mathscr{O}_v)$ $(= H_{\mathscr{O}_v})$. For us, H will be $G_K \cap SL_m(K)$, which satisfies the following two properties:

Strong approximation. H_K is dense in H_A.

Local property. For infinitely many primes w of K, there is an element $\gamma_w \in H(\mathscr{O}_w)$ so that $\det(1-\gamma_w) \neq 0$.

Let $U = \prod_v M_m(\mathscr{O}_v)$, the product over the finite primes of K.

THEOREM 6. *Suppose H as above is a group with the strong approximation and local properties. Suppose $N \in \mathbf{Z}^+$ and $\Gamma_N = H_K \cap (1+NU)$. Then there is an element $\gamma \in \Gamma_N$ with* $\det(1-\gamma) \neq 0$.

PROOF. $H_{\mathbf{A}} \cap (1+NU) = \prod_{v \nmid N} H(\mathscr{O}_v) \cdot \prod_{v|N} (H_v \cap (1+NM_n(\mathscr{O}_v)))$. By the local property, we can find a prime w with $w \nmid N$, and $\gamma_w \in H(\mathscr{O}_w)$ with $\det(1-\gamma_w) \neq 0$. Let $U_w = \{\delta \in H(\mathscr{O}_w) | \det(1-\delta) \neq 0\}$ and let $V = U_w \cdot \prod_{v \neq w} (H_v \cap (1+NM_m(\mathscr{O}_v)))$, a nonempty open subset of $H_{\mathbf{A}}$. By strong approximation, there is an element $\gamma \in V \cap H_K \subset \Gamma_N$. By the choice of V, $\det(1-\gamma) \neq 0$. □

PROOF OF LEMMA 5. Let $H = G_K \cap SL_m(K)$. Choose positive integers c, d, and R so that $d\mathscr{O}_K^m \subset c\mathscr{M} \subset \mathscr{O}_K^m$ and $R\mathscr{N} \subset \mathscr{M}$. Then $\Gamma_{Rd} \subset \Gamma$, so it suffices to prove the lemma for Γ_{Rd}. H has strong approximation by [**3** or **1**], so by Theorem 6 it suffices to show H has the local property. If m is even, choose any w with $w \nmid 2N$, and let $\gamma_w = -1$. If m is odd, choose $\alpha \in K$ so that $\alpha^2 \notin F$. Let $\beta = \alpha/\alpha^\rho$, and choose w so that β is a unit in $\mathscr{O}_w$. Let $\gamma_w = \mathrm{diag}(\beta, \beta, \beta^{-2}, -1, \ldots, -1)$. □

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