ON COMPLEMENTED COPIES OF $c_0$ IN $L^p_X$, $1 \leq p < \infty$

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Let $(S, \Sigma, \mu)$ be a not purely atomic measure space and $X$ be a Banach space. In this note we want to show that if $X$ contains a copy of $c_0$ then the usual Banach space of the Lebesgue-Bochner integrable functions $L^p_X$, $1 \leq p < \infty$, contains a complemented copy of $c_0$. Our result is similar in spirit to one obtained in [1] by Cembranos concerning the Banach space $C_X(K)$; in passing we observe that the Cembranos result has been extended in [3] to the case of $\varepsilon$-tensor products and then in [2] to the case of the Banach space of compact weak*-weak continuous operators.

In order to prove our theorem we need the definition of limited sets. A (bounded) subset $M$ of a Banach space $X$ is said to be limited if for each weak* null sequence $(x_n^*) \subset X^*$ we have $\lim_{n \to \infty} \sup_{x \in M} |x_n^*(x)| = 0$. Further we use the following result obtained in [2].

LEMMA. If $X$ contains an unlimited sequence $(x_n)$ that is equivalent to the unit basis of $c_0$, then $X$ contains a complemented copy of $c_0$.

Now, we are ready to show our theorem.

THEOREM. Assume $X$ contains a copy of $c_0$. Then $L^p_X$, $1 \leq p < \infty$, contains a complemented copy of $c_0$.

PROOF. We shall construct a sequence of functions in $L^p_X$ which is equivalent to the unit basis of $c_0$ and is not limited in $L^p_X$, so by virtue of the Lemma we will be done. Let $(x_n)$ be a sequence in $X$ equivalent to the unit basis of $c_0$ and $(x_n^*)$ be a bounded sequence in $X^*$ such that $x_m^*(x_n) = \delta_{mn}$. It suffices to consider the case of $[0, 1]$ with Lebesgue measure. We consider Rademacher functions $r_n$ and define a sequence $(f_n)$ in $L^p_X$ by putting $f_n = r_n x_n$ and a sequence in $(L^p_X)^*$ by putting $f_n^* = r_n x_n^*$. First of all we show that $(f_n)$ is a sequence equivalent to the unit basis of $c_0$. Since $(x_n)$ is a copy of the unit basis of $c_0$, there are $h_1, h_2 \in \mathbb{R}^+$ such that, for all finite sequences $(a_i)_{i=1}^s$ of real numbers, we have

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i x_i \right\|_X \leq h_2 \max_{1 \leq i \leq s} |a_i|.$$

Since $|r_i(t)| = 1$ on $[0, 1]$ for all $i \in N$, we have

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i r_i(t) x_i \right\|_X \leq h_2 \max_{1 \leq i \leq s} |a_i|, \quad t \in [0, 1].$$

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This easily gives that
\[ h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^{s} a_i f_i \right\|_{L^p_X} \leq h_2 \max_{1 \leq i \leq s} |a_i|, \]
i.e. \((f_n)\) is equivalent to the unit basis of \(c_0\). Now, we observe that
\[ f_n^*(f_n) = \int_{[0,1]} x_n^*(-)(t) r_n^2(t) \, dm = 1 \quad \text{for all } n \in \mathbb{N}. \]
So it remains only to prove that \(f_n^* \rightharpoonup 0\). To this purpose take \(h \in L^p_X\) and observe that
\[ |f_n^*(h)| = \left| \int_{[0,1]} x_n^*(-)(t) r_n(t) \, dm \right| \leq \|x_n^*\| \left\| \int_{[0,1]} h(t) r_n(t) \, dm \right\| \quad \text{for all } n \in \mathbb{N}. \]
Since \((x_n^*)\) is bounded and moreover \(\lim_n \left\| \int_{[0,1]} h(t) r_n(t) \, dm \right\| = 0\), we get \(\lim_n f_n^*(h) = 0\). Arbitrariness of \(h\) in \(L^p_X\) gives that \(f_n^* \rightharpoonup 0\). Our proof is complete.

The above Theorem has the following Corollary.

**Corollary.** Let \(X\) contain a copy of \(c_0\). Then \(L^p_X\), \(1 \leq p < \infty\), is neither a Grothendieck space nor a dual space.

Finally, we want to thank Joe Diestel for suggesting a simplification of our first proof.

**References**

2. G. Emmanuele, A note on Banach spaces containing complemented copies of \(c_0\).

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