A GENERALIZED CONVERSE MEASURABILITY THEOREM
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ABSTRACT. It has long been known that measurability questions in Wiener space and Yeh-Wiener space are often rather delicate. Some converse measurability theorems in Wiener and Yeh-Wiener spaces were proved by Köehler, Skoug, and the first author.

In this paper, we establish a generalized converse measurability theorem by which the above measurability theorems are proved as corollaries.

1. Introduction. Let $C_1[a,b]$ denote the Wiener space of functions of one variable i.e., $C_1[a,b] = \{x(\cdot)|x(a) = 0 \text{ and } x(s) \text{ is continuous on } [a,b]\}$. Let $R = \{(s,t)|a < s < b, \alpha < t < \beta\}$ and let $C_2[R]$ be the Yeh-Wiener space (or 2-parameter Wiener space), i.e., $C_2[R] = \{x(\cdot, \cdot)|x(a,t) = x(s,a) = 0, x(s,t) \text{ is continuous on } R\}$. Let $\nu$ be Wiener measure on $C_1[a,b]$ and let $\lambda$ be Yeh-Wiener measure on $C_2[R]$.

For a discussion of Yeh-Wiener measure see [1, 5, and 6]. Note that Wiener space and Yeh-Wiener space are separable Banach spaces with respect to the supremum norm.

Let $a = t_0 < t_1 < \cdots < t_n = b$ be a subdivision of $[a,b]$. Let $E$ be any subset of Euclidean space $\mathbb{R}^n$ and define $J : C_1[a,b] \to \mathbb{R}^n$ by

$$J(x) = (x(t_1), x(t_2), \ldots, x(t_n)).$$

$J$ is continuous on $C_1[a,b]$ with respect to the uniform topology. By the definition of Wiener measure, if $E$ is Lebesgue measurable then $J^{-1}(E)$ is Wiener measurable and

$$\nu(J^{-1}E) = \int_E g(\vec{\xi}) \, d\gamma(\vec{\xi})$$

where $\gamma$ is Lebesgue measure on $\mathbb{R}^n$,

$$g(\vec{\xi}) = [(2\pi)^n (t_1 - a)(t_2 - t_1) \cdots (t_n - t_{n-1})]^{-1/2} \times \exp\left\{-\frac{1}{2} \sum_{i=1}^{n} (\xi_i - \xi_{i-1})^2 / (t_i - t_{i-1})\right\} > 0,$$

and $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$.

In an unpublished result Köehler established the converse, i.e., if $J^{-1}(E)$ is Wiener measurable then $E$ is Lebesgue measurable. In [5], Skoug extended Köeh-
ler's result to Yeh-Wiener space and established the converse of the one line theorem in [1]. In [2], the first author extended Skoug's result to the n-parallel lines theorem in [1]. Later the proof of the converse of the n-parallel lines theorem was simplified [3].

In this paper, we establish a generalized converse measurability theorem which has the above converse measurability theorems as corollaries. The techniques used in this paper are quite different from the techniques used in [2, 3, and 5].

2. A generalized converse measurability theorem. A probability measure $P$ on a $\sigma$-algebra $\mathcal{S}$ containing the Borel sets in a topological space is called tight if for any $\varepsilon > 0$ and for any $E$ in $\mathcal{S}$ there exists a compact set $K \subset E$ such that $P(F \setminus K) < \varepsilon$. It is known that any probability measure on the Borel class of a complete separable metric space is tight [4].

Let $P$ be a tight measure on $\mathcal{B}(S)$ and let $m$ be a measure on $\mathcal{B}(T)$ where $\mathcal{B}(S)$ and $\mathcal{B}(T)$ are the Borel classes of topological spaces $S$ and $T$, respectively. Let $(S, \mathcal{B}(S), P)$ and $(T, \mathcal{B}(T), m)$ be the completions of $(S, \mathcal{B}(S), P)$ and $(T, \mathcal{B}(T), m)$, respectively. It is easy to see that $P$ is tight on $\mathcal{B}(S)$.

Let $J: S \to T$ be continuous and let $\mathcal{U} = \{ E \subset T: J^{-1}(E) \text{ is } P\text{-measurable} \}$. Define a set function $\mu$ on $\mathcal{U}$ by $\mu(E) = P(J^{-1}(E))$ where $E \in \mathcal{U}$. $\mathcal{U}$ is a $\sigma$-algebra which contains the Borel sets in $T$ as is easily checked and $(T, \mathcal{U}, \mu)$ is a complete tight measure space. To see that $\mu$ is tight on $\mathcal{U}$, let $E$ be $\mu$-measurable. Then for each positive integer $n$, there exists a compact subset $K_n$ of $J^{-1}(E)$ such that $P(J^{-1}(E) \setminus K_n) < 1/n$. Put $C_n = \bigcup_{k=1}^{n} K_k$. Then $P(J^{-1}(E) \setminus C_n) < 1/n$. Since $C_n$ is compact, $J(C_n)$ is compact and

$$\mu(E \setminus J(C_n)) = P(J^{-1}(E \setminus J(C_n))) \leq P(J^{-1}(E) \setminus C_n) < 1/n.$$ 

Hence $\mu$ is tight on $\mathcal{U}$.

**Lemma 2.1.** $\mathcal{U} = \mathcal{B}(T)$ under the following assumption: $N$ is an $m$-null set if and only if $N$ is a $\mu$-null set.

**Proof.** It is easy to see that $\mathcal{B}(T) \subset \mathcal{U}$. To show that $\mathcal{U} \subset \mathcal{B}(T)$, let $E \in \mathcal{U}$. Since $\mu$ is tight, there exists a compact set $K_n$ such that $K_n \subset E$ and $\mu(E \setminus K_n) < 1/n$ for each $n = 1, 2, \ldots$. Let $K = \bigcup_{n=1}^{\infty} K_n$. Then $K$ is a Borel set and $K \subset E$. Since $\mu(E \setminus K) \leq \mu(E \setminus K_n) < 1/n$ for each $n = 1, 2, \ldots, \mu(E \setminus K) = 0$. By assumption, $E \setminus K$ is an $m$-null set. Then $E = K \cup (E \setminus K)$ is $m$-measurable and so $\mathcal{U} \subset \mathcal{B}(T)$.

For the measurability theorems in Wiener and Yeh-Wiener spaces in which we are concerned, we have $\mathcal{B}(T) \subset \mathcal{U}$. For example, if $E$ is Lebesgue measurable then $J^{-1}(E)$ is Wiener measurable. To establish the converse measurability theorems we need $\mathcal{U} = \mathcal{B}(T)$.

**Theorem 2.2.** Suppose $\mathcal{B}(T) \subset \mathcal{U}$. If there exists an integrable function $g: T \to \mathbb{R}$ such that $g(t) > 0$ for $m$-a.e. $t$ and $\mu E = \int_E g(t) \, d\tilde{m}(t)$ for every $m$-measurable set $E$, then $\mathcal{U} = \mathcal{B}(T)$.

**Proof.** By Lemma 2.1, it suffices to show that if $N$ is a $\mu$-null set then $N$ is an $m$-null set. Now we assume that $N$ is a $\mu$-null set and it is not an $m$-null set. We consider two cases: (1) $N$ is $m$-measurable, (2) $N$ is not $m$-measurable.
If $N$ is $\overline{m}$-measurable, then there exists a Borel set $B \subset N$ such that $mB = \overline{m}N > 0$. Since $\mu B = \int_B g(t) \, d\overline{m}(t) > 0$, $\mu_N > 0$ where $\mu_*$ is the inner measure of $\mu$. Since $\mu N = \mu_* N$, $\mu N > 0$ which is a contradiction.

If $N$ is not $\overline{m}$-measurable, i.e., $N \notin \mathcal{B}(\mathcal{T})$, then $mB > 0$ for every Borel set $B \supset N$. Then $\mu B > 0$ for any Borel set $B \supset N$. Since $\mu$ is tight, there exists a compact set $K_n \subset N^c$ such that $\mu(N^c \setminus K_n) < 1/n$ for each $n$. Let $K = \bigcup_{n=1}^\infty K_n$. Then $K^c$ is a Borel set and $N \subset K^c$.

$$
\mu(K^c) = 1 - \mu(K) = \mu(N^c) - \mu(K) \leq \mu(N^c \setminus K_n) < 1/n \quad \text{for each } n.
$$

Hence $\mu(K^c) = 0$. This contradicts the fact that $\mu B > 0$ for any Borel set $B \supset N$. Therefore we conclude that every $\mu$-null set is also an $\overline{m}$-null set.

3. Converse measurability theorems.

**THEOREM 3.1 (KÖHLE).** Let $a = t_0 < t_1 < \cdots < t_n = b$. Let $E$ be any subset of $\mathbb{R}^n$ and let $J$ be defined as in (1). Then $E$ is Lebesgue measurable if and only if $J^{-1}(E)$ is Wiener measurable.

**PROOF.** By the definition of Wiener measure, if $E$ is Lebesgue measurable, then $J^{-1}(E)$ is Wiener measurable. To show the converse, let

$$
\mathcal{U} = \{E \subset \mathbb{R}^n | J^{-1}(E) \text{ is Wiener measurable}\}.
$$

Then $\mathcal{U}$ contains all Lebesgue measurable sets in $\mathbb{R}^n$. For $E \in \mathcal{U}$, we define $\mu$ on $\mathcal{U}$ by $\mu E = \nu(J^{-1}(E))$. Then every Lebesgue measurable set $E$ satisfies (2). Since Wiener measure is tight on the $\sigma$-algebra of Wiener measurable sets which contains the Borel sets in $C_1[a,b], (\mathbb{R}^n, \mathcal{U}, \mu)$ is a complete tight measure space. Hence by Theorem 2.2, $J^{-1}(E)$ is Wiener measurable if and only if $E$ is Lebesgue measurable.

Let $a = s_0 < s_1 < \cdots < s_m = b, \alpha = t_0 < t_1 < \cdots < t_n = \beta$. Let $E$ be any subset of $\mathbb{R}^{mn}$ and define $J: C_2[R] \to \mathbb{R}^{mn}$ by

$$
J(x) = (x(s_1, t_1), \ldots, x(s_m, t_n)).
$$

$J$ is continuous on $C_2[R]$ with respect to the uniform topology. By the definition of Yeh-Wiener measure, if $E$ is Lebesgue measurable then $J^{-1}(E)$ is Yeh-Wiener measurable and

$$
\lambda(J^{-1}(E)) = \int_E g(\tilde{\xi}) \, d\gamma(\tilde{\xi})
$$

for any Lebesgue measurable set $E$ in $\mathbb{R}^{mn}$, where $\gamma$ is Lebesgue measure on $\mathbb{R}^{mn}$,

$$
g(\tilde{\xi}) = \{(2\pi)^mn \{((s_1 - a) \cdots (s_m - s_{m-1})(t_1 - \alpha) \cdots (t_n - t_{n-1})\}^{-1/2}
\times \exp \left\{ -\frac{1}{2} \sum_{k=1}^n \sum_{j=1}^m (\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})^2 / (s_j - s_{j-1})(t_k - t_{k-1}) \right\}
> 0,
$$

and $\tilde{\xi} = (\xi_{1,1}, \xi_{1,2}, \ldots, \xi_{m,n}) \in \mathbb{R}^{mn}$.

We will omit the proof of Theorem 3.2 below since it is similar to the proof given above for Theorem 3.1.
THEOREM 3.2 (SKOUG). Let $J$ be defined as in (3) and let $E$ be any subset of $\mathbb{R}^{mn}$. Then $E$ is Lebesgue measurable if and only if $J^{-1}(E)$ is Yeh-Wiener measurable.

We shall use the following notation for the Cartesian product of $n$ Wiener spaces $\hat{X}C_1[a, b] = C_1[a, b] \times \cdots \times C_1[a, b]$ and $\hat{X}\nu = \nu \times \cdots \times \nu$ will denote the product of $n$ Wiener measures on $\hat{X}C_1[a, b]$.

Let $t_0 = t_1 < \cdots < t_n = \beta$ be a subdivision of $[\alpha, \beta]$. Define $\psi: \hat{X}C_1[a, b] \to \hat{X}C_1[a, b]$ by

$$\psi(y_1, y_2, \ldots, y_n) = \left( \sqrt{\frac{t_1 - t_0}{2}} y_1, \sqrt{\frac{t_1 - t_0}{2}} y_2, \ldots, \sqrt{\frac{t_2 - t_1}{2}} y_1 + \frac{\sqrt{t_2 - t_1}}{2} y_2, \ldots, \sqrt{\frac{t_n - t_{n-1}}{2}} y_1 + \cdots + \sqrt{\frac{t_n - t_{n-1}}{2}} y_n \right).$$

Then $\psi$ is 1-1 and onto. $\psi$ and $\psi^{-1}$ are continuous with respect to the uniform topology. Let $G: C_2[R] \to \hat{X}C_1[a, b]$ be defined by $G(x) = (x(\cdot, t_1), \ldots, x(\cdot, t_n))$. Then $G$ is a continuous function from $C_2[R]$ onto $\hat{X}C_1[a, b]$. In [1] Cameron and Storvick evaluated certain Yeh-Wiener integrals in terms of Wiener integrals. In particular they obtained the $n$-parallel lines theorem. The converse of the $n$-parallel lines theorem follows quite easily once Theorem 3.3 below is established [2].

THEOREM 3.3. Let $A$ be any subset of $\hat{X}C_1[a, b]$. Then $\psi^{-1}A$ is $\hat{X}\nu$-measurable if and only if $G^{-1}A$ is Yeh-Wiener measurable. Furthermore,

$$\lambda(G^{-1}A) = \hat{X}\nu(\psi^{-1}A).$$

PROOF. By the $n$-parallel lines theorem, if $\psi^{-1}A$ is $\hat{X}\nu$-measurable then $G^{-1}A$ is Yeh-Wiener measurable and $\lambda(G^{-1}A) = \hat{X}\nu(\psi^{-1}A)$ [2]. To show the converse, let $J = \psi^{-1} \circ G$. Then $J$ is a continuous function from $C_2[R]$ onto $\hat{X}C_1[a, b]$. Let

$$\mathcal{U} = \{ E \in \hat{X}C_1[a, b]: J^{-1}(E) \text{ is Yeh-Wiener measurable} \}.$$

Define a set function $\mu$ on $\mathcal{U}$ by $\mu E = \lambda(J^{-1}E))$. For any $\hat{X}\nu$-measurable set $E = \psi^{-1}(\psi E)$, $G^{-1}(\psi E) = J^{-1}(E)$ is Yeh-Wiener measurable and $\lambda(G^{-1}\psi E) = \hat{X}\nu(\psi^{-1}\psi E) = \hat{X}\nu(E)$ by the $n$-parallel lines theorem. Hence $E \in \mathcal{U}$ and $\mu E = \lambda(J^{-1}E) = \lambda(G^{-1}\psi E) = \hat{X}\nu(E) = \int_E 1 \, d \hat{X}\nu$. Hence by Theorem 2.2, $\mathcal{U}$ is the $\sigma$-algebra of $\hat{X}\nu$-measurable sets. If $G^{-1}A = J^{-1}(\psi^{-1}A)$ is Yeh-Wiener measurable, then $\psi^{-1}A \in \mathcal{U}$ and so $\psi^{-1}A$ is $\hat{X}\nu$-measurable.

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References


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