ABSTRACT. We show that one of the binomial numbers discovered by K. J. Huang provides an example of topologically equivalent measures in $2^\mathbb{N}$ that are not trivially homeomorphic.

Two Borel measures $\mu$ and $\nu$ in a topological space $X$ are said to be homeomorphic, or topologically equivalent, if $\mu = \nu h$ for some homeomorphism $h : X \to X$. In the Cantor space $X = \{0,1\}^\mathbb{N}$, for any $0 \leq r \leq 1$ let $\mu(r) = r^{\mu_n}$, where $\mu_n(0) = r$ and $\mu_n(1) = 1 - r$ for all $n$. $\mu(1 - r)$ is always homeomorphic to $\mu(r)$, by the mapping $R$ that interchanges 0 and 1 in each coordinate space. When $r \in [0,1]$ is rational, transcendental, or an algebraic integer of degree 2, it is known that no other member of $\{\mu(p) : 0 \leq p \leq 1\}$ is homeomorphic to $\mu(r)$. This is because a number $s \in [0,1]$ is binomially related to such a number $r$ if and only if $s = r$, or $s = 1 - r$ [2, 1]. For each $n > 2$ Huang [1] exhibited an algebraic integer $r \in (0,1)$ (and also a noninteger) of degree $n$ that is binomially related to at least one number $s \neq r$, $s \neq 1 - r$. Pinch [4] showed that in case $n = 2k + 1$ there are at least $2k$ such numbers $s$. However, it has remained an open question whether $\mu(s)$ is homeomorphic to $\mu(r)$ in any of these cases. We shall show that the measures corresponding to Huang's algebraic integer of degree 3 are, in fact, homeomorphic. The other cases remain open.

**Theorem.** Let $r$ be the unique real root of the equation $r^3 + r^2 - 1 = 0$ and let $s = r^2$. Then $\mu(s)$ is homeomorphic to $\mu(r)$, as well as to $\beta(1 - r)$ and $\beta(1 - s)$.

For any $u \in \{0,1\}^n$, $n \in \mathbb{N}$, the set $\langle u \rangle$ of points of $X$ whose first $n$ coordinates coincide with $u$ is called a thin cylinder.

**Lemma.** Let $U_i = \langle u_i \rangle$ and $V_i = \langle v_i \rangle$ ($i = 1,2,3$) be two indexed partitions of $X$ into three thin cylinders, and suppose that

$$\mu(q)(U_i) = \mu(p)(V_i) \quad (i = 1,2,3).$$

Then $\mu(q) = \mu(p)h$ for some homeomorphism $h : X \to X$.

This is a consequence of a general sufficient condition [3, Theorem 3.1] for the existence of a homeomorphism between shift-invariant measures in different spaces of the form $\{1,2,\ldots,k\}^\mathbb{N}$. It may be proved directly as follows.

$X$ can be partitioned into three thin cylinders in only two ways, so $U_i$ and $V_i$ must be indexings of either $\{(0),(1,0),(1,1)\}$ or $\{(1),(0,0),(0,1)\}$. Each $x \in X$
can be written uniquely as a sequence of blocks \( b_1, b_2, \ldots \) equal to \( u_1, u_2, \) or \( u_3 \). Let \( h(x) \) be the element of \( X \) defined by the sequence of blocks \( b'_1, b'_2, \ldots \), where \( b'_i = v_1, v_2, \) or \( v_3 \), according as \( b_i = u_1, u_2, \) or \( u_3 \). Evidently \( h \) is bijective. The class \( \mathcal{B} \) of thin cylinders of the form \( (b_1, b_2, \ldots, b_n) \) constitute a base, and so does \( h(\mathcal{B}) \), because every thin cylinder is the union of at most two members of either class. Hence \( h \) is a homeomorphism. Equations (1) and the definition of product measure imply that \( \mu(q)(B) = \mu(p)(h(B)) \) for each \( B \in \mathcal{B} \). Since every open set is a countable disjoint union of members of \( \mathcal{B} \) it follows that \( \mu(q) = \mu(p)h \).

If we take
\[
U_1 = \langle 0 \rangle, \quad U_2 = \langle 1, 0 \rangle, \quad U_3 = \langle 1, 1 \rangle
\]
and
\[
V_1 = \langle 0, 0 \rangle, \quad V_2 = \langle 1 \rangle, \quad V_3 = \langle 0, 1 \rangle,
\]
then equations (1) become
\[
q = p^2, \quad q(1 - q) = 1 - p, \quad (1 - q)^2 = p(1 - p),
\]
which are satisfied by \( p = r, q = s \). This completes the proof of the theorem. The corresponding homeomorphism \( h \) leaves \( (1, 0, 0) \) invariant and has four fixed points.

Alternatively, we could take \( V_2, V_3, V_1 \) in place of \( U_1, U_2, U_3 \). Then equations (1) become
\[
1 - q = p^2, \quad q(1 - q) = 1 - p, \quad q^2 = p(1 - p),
\]
which are satisfied by \( p = r, q = 1 - s \). In this case the corresponding homeomorphism permutes \( V_3, V_2, V_1 \) cyclically and has period 3. It is equal to \( hR \).

It should be noted, however, that \( h \) is never unique; if \( \mu(q) = \mu(p)h \), then \( h \) can always be replaced by \( hg \) or \( gh \), where \( g \) is an arbitrary permutation of the coordinate spaces.

It is easy to verify that all possible choices of \( U_i \) and \( V_i \) lead to equations (1) that have no solutions other than ones for which \( q = p, q = 1 - p \), or for which \( \{p, q\} \subset \{1 - r, 1 - s, s, r\} \), so no further equivalences can be obtained from the lemma as stated.

REFERENCES


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