PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 3, November 1988

A CHARACTERIZATION OF COMPLETE INTERSECTION CURVES IN $\mathbf{P}^3$

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. We prove the following theorem. Let $C \subset \mathbf{P}^3$ be a reduced and irreducible curve not lying on a quadric. If the generic plane section Γ of C is a complete intersection then C is a complete intersection.

Introduction. Let k be an algebraically closed field of characteristic 0 and let $\mathbf{P}^3 = \mathbf{P}^3_k$. Let $C \subset \mathbf{P}^3$ be a reduced and irreducible curve of degree d and let $\Gamma = C \cap H$ be its generic plane section.

It is known (see [**H**]) that Γ consists of d distinct points of H with the uniform position property.

Now suppose that $d = st$, $t \geq s$, and that Γ is a complete intersection of two curves of H of degrees s and t (shortly Γ is a c.i. (s,t)). We will use a theorem of Laudal (see [**L**]) which says that, if $t \geq s+1$, then C lies on a surface of degree s. In particular if $s = 2$ then C lies on a quadric and, conversely, every curve C of even degree $d = 2t$ on a quadric is such that Γ is a c.i. $(2,t)$.

We will prove, in §2, that, if $s \geq 3$, then C is a complete intersection of two surfaces of degrees s and t (shortly C is a c.i. (s,t)).

1. First we recall a definition.

DEFINITION 1. Let $\Gamma \subset \mathbf{P}^2$ be a set of d points. We say that Γ has the uniform position property (UPP for short) if, for every subset Γ' of d' points of Γ and any n, it is

$$h_\Gamma(n) = \min(d', h_\Gamma(n)),$$

where h_Γ is the Hilbert function of Γ.

THEOREM 2. *Let Γ be the generic plane section of a reduced and irreducible curve $C \subset \mathbf{P}^3$ of degree d. Then Γ has the UPP.*

PROOF. See [**H**, §2].

For a set $\Gamma \subset \mathbf{P}^2$ of d points define:
s = least degree of a curve $F \subset \mathbf{P}^2$ containing Γ,
t = least degree of a curve $G \subset \mathbf{P}^2$ containing Γ and not containing F.

We have the following

PROPOSITION 3. *Let $\Gamma \subset \mathbf{P}^2$ be a set of points with the UPP and let s,t be as above. Then Γ is contained in a complete intersection (s,t).*

PROOF. See [**M-R**, Proposition 1].

Received by the editors June 26, 1987 and, in revised form, November 16, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14H45, 14M10; Secondary 14H50.
Work done with the financial support of M.P.I.

THEOREM 4. *Let $C \subset \mathbf{P}^3$ be a reduced and irreducible curve of degree d and Γ its generic plane section. Let s be as above.*

If $d \geq s(s+1)$, then C lies on a surface of degree s.

PROOF. In the case $d > s(s+1)$ see [**L**, Corollary, p. 147]. If $d = s(s+1)$ by Proposition 3 two cases are possible: either $t > s+1$ or $t = s+1$ and in the second case Γ is a c.i. (s,t). In both cases Corollary p. 145 of [**L**] applies.

REMARK 5. From Proposition 3 it follows easily that, if Γ has the UPP and its Hilbert function is the Hilbert function of a c.i. (s,t), then actually Γ is a c.i. (s,t).

2. In this section we prove the following

THEOREM 6. *Let $C \subset \mathbf{P}^3$ be a reduced and irreducible curve of degree d not lying on a quadric. If the generic plane section Γ is a c.i. (s,t), then C is a c.i. (s,t).*

PROOF. *Case* 1. C lies on a surface of degree s.

This, by Theorem 4, is the case when $t \geq s+1$.

Suppose C is not a c.i. (s,t). For a plane H consider the exact sequence

$$\begin{aligned} H^0(\mathscr{I}_C(t-1)) \to H^0(\mathscr{I}_C(t)) \to H^0(\mathscr{I}_\Gamma(t)) \to H^1(\mathscr{I}_C(t-1)) \\ \to H^1(\mathscr{I}_C(t)) \to H^1(\mathscr{I}_\Gamma(t)). \end{aligned}$$

For a generic H, the map

$$\varphi_H \colon H^1(\mathscr{I}_C(t-1)) \to H^1(\mathscr{I}_C(t))$$

has a kernel of dimension 1.

Claim. if $\alpha \in H^1(\mathscr{I}_C(t-1))$, $\alpha \neq 0$, is such that $\alpha H = 0$, then $\alpha H' \in \operatorname{Im}\varphi_H$ for every H'.

In fact let $\alpha_1, \ldots, \alpha_h = \alpha$ be a basis of $H^1(\mathscr{I}_c(t-1))$. Assume that $\alpha H' \notin \operatorname{Im}\varphi_H$. We can assume that $\alpha_1 H', \ldots, \alpha_v H'$ generate $V = \operatorname{Im}\varphi_H \cap \operatorname{Im}\varphi_{H'}$ and $\alpha_{v+1}H', \ldots, \alpha_h H'$ are linearly independent and generate a subspace $S \subset \operatorname{Im}\varphi_{H'}$ such that $S \cap V = (0)$.

Now consider the plane $H + \lambda H'$; it is easy to see that, for generic λ, the map $\varphi_{H+\lambda H'}$ has rank $h-1$ and $\alpha_1(H+\lambda H'), \ldots, \alpha_{h-1}(H+\lambda H')$ are linearly independent: this is done by looking at the $(h-1)\times(h-1)$ minors of the matrix associated to $\varphi_{H+\lambda H'}$.

Hence we have a linear relation

$$\sum_{j=1}^{h} a_i \alpha_i (H + \lambda H') = 0.$$

But since $\alpha_1(H+\lambda H'), \ldots, \alpha_v(H+\lambda H') \in \operatorname{Im}\varphi_H$ it follows that $a_{v+1} = \cdots = a_h = 0$. Hence we have a nontrivial relation

$$\sum_{j=1}^{v} a_i \alpha_i (H + \lambda H') = 0$$

which is absurd.

Now take four linearly independent planes $H_0 = H$, H_1, H_2, H_3.

The fact that $\alpha H_i \in \operatorname{Im}\varphi_H$ $(i = 1,2,3)$ means that, if α_Γ is the image of α in $H^1(\mathscr{I}_\Gamma(t-1))$ and $L_i = H_i \cap H$, we have $\alpha_\Gamma L_i = 0$ in $H^1(\mathscr{I}_\Gamma(t))$.

Let $\sigma \in H^0(\mathscr{O}_C(t-1))$ be such that the image of σ in $H^1(\mathscr{I}_C(t-1))$ is α and let σ_Γ be the restriction of σ to Γ. We have

$$\sigma_\Gamma L_i \in H^0(\mathscr{O}_H(t)).$$

We identify H with $\mathbf{P}^2$ and let $R = k[X_1, X_2, X_3]$ be the polynomial ring; we can suppose $L_i = X_i$, $i = 1,2,3$.

Then there are forms $Q_i \in R_t$ such that $\sigma_\Gamma X_i = Q_i$ in $H^0(\mathscr{O}_\Gamma(t))$. From this it follows that $Q_i X_j = Q_j X_i \bmod H^0(\mathscr{I}_\Gamma(t+1))$, for all $i,j = 1,2,3$.

We use the following

LEMMA. *With the notation as before let $J \subset R$ be the homogeneous ideal of Γ i.e. $J = \sum_{n=0}^{\infty} H^0(\mathscr{I}_\Gamma(n))$. Then the following are equivalent:*

(1) $\operatorname{Tor}_1^R(J,k)_{t+2} = 0$,

(2) *for every triple (Q_1, Q_2, Q_3), $Q_i \in R_t$, $i = 1,2,3$, such that*

$$X_i Q_j - X_j Q_i \in J_{t+1}$$

for all $i,j = 1,2,3$, there exists $Q \in R_{t-1}$ such that

$$Q_i - X_i Q \in J_t, \ i = 1,2,3.$$

PROOF. See [**L**, Lemma, p. 141].

We observe that $\dim \operatorname{Tor}_1^R(J,k)_{t+2}$ is the number of sizygies of J in degree $t+2$: this is easily seen by taking a free resolution of the R-module J and tensoring by k.

In our case, since Γ is a c.i. (s,t) and $s \geq 3$, $\operatorname{Tor}_1^R(J,k)_{t+2} = 0$, hence by the lemma we have

$$\sigma_\Gamma X_i = Q X_i \text{ in } H^0(\mathscr{O}_\Gamma(t));$$

but we can assume that X_i does not vanish on any point of Γ, hence $\sigma_\Gamma = Q$ in $H^0(\mathscr{O}_\Gamma(t-1))$.

This implies that the image of α is zero $H^1(\mathscr{I}_\Gamma(t-1))$, hence there exist $\beta \in H^1(\mathscr{I}_C(t-2))$ such that $\alpha = \beta H$.

So we have that, for a generic plane H, the map

$$\varphi_{H^2} \colon H^1(\mathscr{I}_C(t-2)) \to H^1(\mathscr{I}_C(t))$$

has a kernel of dimension 1, since the map $H^1(\mathscr{I}_C(t-2)) \to H^1(\mathscr{I}_C(t-1))$ is injective.

Let us continue, assuming by induction that the map

$$\varphi_{H^l} \colon H^1(\mathscr{I}_C(t-l)) \to H^1(\mathscr{I}_C(t))$$

has, for a generic H, a kernel of dimension 1 and prove that $\varphi_{H^{l+1}}$ has the same property. This proves the theorem since, for $l = t$, it is $H^1(\mathscr{I}_C) = 0$.

Suppose $\alpha H^l = 0$ and prove first that $\alpha H'^l \in \operatorname{Im}\varphi_H$ for any H'. The proof is as before; let $\alpha_1, \ldots, \alpha_m = \alpha$ be a basis of $H^1(\mathscr{I}_C(t-l))$ and assume that $\alpha H'^l \notin \operatorname{Im}\varphi_H$. We can assume that $\alpha_1 H'^l, \ldots, \alpha_v H'^l$ generate $V = \operatorname{Im}\varphi_H \cap \operatorname{Im}\alpha_{H'^l}$ and $\alpha_{v+1} H'^l, \ldots, \alpha_m H'^l$ are linearly independent and generate a subspace S such that $S \cap V = (0)$.

Consider $(H+\lambda H')^l$: for λ generic, $\varphi_{(H+\lambda H')^l}$ has rank $m-1$ and $\alpha_1(H+\lambda H')^l, \dots, \alpha_{m-1}(H+\lambda H')^l$ are linearly independent.

We have a relation

$$\sum_{j=1}^{m} a_i\alpha_i(H+\lambda H')^l = 0$$

but since $\alpha_i(H+\lambda H')^l, \dots, \alpha_v(H+\lambda H')^l \in \operatorname{Im}\varphi_H$ it follows that $a_{v+1} = \cdots = a_m = 0$.

Hence we have a nontrivial relation

$$\sum_{j=1}^{v} a_i\alpha_i(H+\lambda H')^l = 0$$

which is absurd.

Now we consider the planes $H_0 = H, H_1, H_2, H_3$ and the plane $H_1+\lambda H_2+\mu H_3$; we have $\alpha(H_1+\lambda H_2+\mu H_3)^l \in \operatorname{Im}\varphi_H$ for every λ, μ.

Developing the power we get that

$$\alpha H_{i_1}H_{i_2}\cdots H_{i_l} \in \operatorname{Im}\varphi_H$$

for every $1 \le i_1 \le i_2 \le \cdots \le i_l \le 3$.

Using the lemma we get $\alpha = \beta H$ with $\beta \in H^1(\mathscr{I}_c(t-l-1))$.

Case 2. $d = s^2$ and C does not lie on a surface of degree s.

In this case the map

$$\varphi_H\colon H^1(\mathscr{I}_C(s-1)) \to H^1(\mathscr{I}_C(s))$$

has, for a generic H, a kernel of dimension 2.

Let $\alpha, \bar\alpha$ be a basis of $\ker\varphi_H$. We prove that, for every H', $\alpha H' \in \operatorname{Im}\varphi_H$ and $\bar\alpha H' \in \operatorname{Im}\varphi_H$ from which the theorem follows in a similar way as in case 1.

Let $V = \operatorname{Im}\varphi_H \cap \operatorname{Im}\varphi_{H'}$ and let W be the subspace generated by $\alpha H', \bar\alpha H'$.

(a) Suppose first that $V \cap W = (0)$.

Let $\alpha_1, \dots, \alpha_{h-1} = \alpha$, $\alpha_h = \bar\alpha$ be a basis of $H^1(\mathscr{I}_C(s-1))$; we can assume that $\alpha_1 H', \dots, \alpha_v H' \in V$ and $\alpha_{v+1}H', \dots, \alpha_{h-1}H'$, $\alpha_h H'$ are linearly independent and generate a subspace S such that $S \cap V = (0)$.

We obtain a relation, for a generic λ,

$$\sum_{j=1}^{v} a_i\alpha_i(H+\lambda H') = 0$$

which is absurd.

(b) Suppose $\dim V \cap W = 1$. We can assume $\alpha H' \in V$, $\bar\alpha H' \notin V$.

We can choose a basis $\alpha = \alpha_1, \dots, \alpha_h = \bar\alpha$ of $H^1(\mathscr{I}_c(s-1))$ such that $\alpha_1 H', \dots, \alpha_v H' \in V$ but $\alpha_{v+1}H', \dots, \alpha_h H'$ are linearly independent and generate a subspace S such that $S \cap V = (0)$.

We have two relations

$$\sum_{j=1}^{v} a_i\alpha_i(H+\lambda H') = 0, \quad \sum_{j=1}^{v} b_i\alpha_i(H+\lambda H') = 0$$

from which a nontrivial relation $\sum_{i=2}^{v} c_i\alpha_i(H+\lambda H') = 0$ which is absurd.

REMARK 7. Theorem 6 holds also in the case when $\mathscr{k}$ has characteristic $p > 0$, provided p is sufficiently high; precisely p must be greater than the coefficients involved in the power $(H_1 + \lambda H_2 + \mu H_3)^l$, $1 \leq l \leq t - 1$.

REMARK 8. With the same proof of Theorem 6, case 1 (with s instead of t) one can prove that, if Γ is a c.i. (s, t), $t \geq s + 1$, then C lies on a surface of degree s. In this way one can avoid the use of Laudal's theorem.

REMARK 9. The same idea of the proof of Theorem 6 can be used to give an alternative and simpler proof of Laudal's Theorem, avoiding the use of deformation theory (see [S] for details).

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