A CHARACTERIZATION OF COMPLETE INTERSECTION CURVES IN $\mathbb{P}^3$

ROSARIO STRANO

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ABSTRACT. We prove the following theorem. Let $C \subset \mathbb{P}^3$ be a reduced and irreducible curve not lying on a quadric. If the generic plane section $\Gamma$ of $C$ is a complete intersection then $C$ is a complete intersection.

Introduction. Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 and let $\mathbb{P}^3 = \mathbb{P}^3$. Let $C \subset \mathbb{P}^3$ be a reduced and irreducible curve of degree $d$ and let $\Gamma = C \cap H$ be its generic plane section.

It is known (see [H]) that $\Gamma$ consists of $d$ distinct points of $H$ with the uniform position property.

Now suppose that $d = st$, $t \geq s$, and that $\Gamma$ is a complete intersection of two curves of $H$ of degrees $s$ and $t$ (shortly $\Gamma$ is a c.i. $(s, t)$). We will use a theorem of Laudal (see [L]) which says that, if $t > s + 1$, then $C$ lies on a surface of degree $s$. In particular if $s = 2$ then $C$ lies on a quadric and, conversely, every curve $C$ of even degree $d = 2t$ on a quadric is such that $\Gamma$ is a c.i. $(2, t)$.

We will prove, in §2, that, if $s \geq 3$, then $C$ is a complete intersection of two surfaces of degrees $s$ and $t$ (shortly $C$ is a c.i. $(s, t)$).

1. First we recall a definition.

DEFINITION 1. Let $T \subset \mathbb{P}^2$ be a set of $d$ points. We say that $T$ has the uniform position property (UPP for short) if, for every subset $T'$ of $d'$ points of $T$ and any $n$, it is

$$h_T(n) = \min(d', h_T(n)),$$

where $h_T$ is the Hilbert function of $\Gamma$.

THEOREM 2. Let $T$ be the generic plane section of a reduced and irreducible curve $C \subset \mathbb{P}^3$ of degree $d$. Then $T$ has the UPP.

PROOF. See [H, §2].

For a set $T \subset \mathbb{P}^2$ of $d$ points define:

$s = \text{least degree of a curve } F \subset \mathbb{P}^2 \text{ containing } T$,

t = \text{least degree of a curve } G \subset \mathbb{P}^2 \text{ containing } T \text{ and not containing } F$.

We have the following

PROPOSITION 3. Let $\Gamma \subset \mathbb{P}^2$ be a set of points with the UPP and let $s, t$ be as above. Then $\Gamma$ is contained in a complete intersection $(s, t)$.

PROOF. See [M-R, Proposition 1].
THEOREM 4. Let \( C \subset \mathbb{P}^3 \) be a reduced and irreducible curve of degree \( d \) and \( \Gamma \) its generic plane section. Let \( s \) be as above.

If \( d \geq s(s+1) \), then \( C \) lies on a surface of degree \( s \).

PROOF. In the case \( d > s(s+1) \) see [L, Corollary, p. 147]. If \( d = s(s+1) \) by Proposition 3 two cases are possible: either \( t > s+1 \) or \( t = s+1 \) and in the second case \( \Gamma \) is a c.i. \((s,t)\). In both cases Corollary p. 145 of [L] applies.

REMARK 5. From Proposition 3 it follows easily that, if \( \Gamma \) has the UPP and its Hilbert function is the Hilbert function of a c.i. \((s,t)\), then actually \( \Gamma \) is a c.i. \((s,t)\).

2. In this section we prove the following

THEOREM 6. Let \( C \subset \mathbb{P}^3 \) be a reduced and irreducible curve of degree \( d \) not lying on a quadric. If the generic plane section \( \Gamma \) is a c.i. \((s,t)\), then \( C \) is a c.i. \((s,t)\).

PROOF. Case 1. \( C \) lies on a surface of degree \( s \).

This, by Theorem 4, is the case when \( t > s+1 \).

Suppose \( C \) is not a c.i. \((s,t)\). For a plane \( H \) consider the exact sequence

\[
\begin{align*}
H^0(\mathcal{I}_C(t-1)) &\to H^0(\mathcal{I}_C(t)) \to H^0(\mathcal{I}_T(t)) \to H^1(\mathcal{I}_C(t-1)) \\
&\to H^1(\mathcal{I}_C(t)) \to H^1(\mathcal{I}_T(t)).
\end{align*}
\]

For a generic \( H \), the map

\[
\varphi_H : H^1(\mathcal{I}_C(t-1)) \to H^1(\mathcal{I}_C(t))
\]

has a kernel of dimension 1.

Claim. if \( \alpha \in H^1(\mathcal{I}_C(t-1)) \), \( \alpha \neq 0 \), is such that \( \alpha H = 0 \), then \( \alpha H' \in \text{Im} \varphi_H \) for every \( H' \).

In fact let \( \alpha_1, \ldots, \alpha_h = \alpha \) be a basis of \( H^1(\mathcal{I}_C(t-1)) \). Assume that \( \alpha H' \notin \text{Im} \varphi_H \). We can assume that \( \alpha_1 H', \ldots, \alpha_v H' \) generate \( V = \text{Im} \varphi_H \cap \text{Im} \varphi_{H'} \) and \( \alpha_{v+1} H', \ldots, \alpha_h H' \) are linearly independent and generate a subspace \( S \subset \text{Im} \varphi_{H'} \) such that \( S \cap V = \{0\} \).

Now consider the plane \( H + \lambda H' \); it is easy to see that, for generic \( \lambda \), the map \( \varphi_{H+\lambda H'} \) has rank \( h - 1 \) and \( \alpha_1(H + \lambda H'), \ldots, \alpha_{h-1}(H + \lambda H') \) are linearly independent: this is done by looking at the \((h-1) \times (h-1)\) minors of the matrix associated to \( \varphi_{H+\lambda H'} \).

Hence we have a linear relation

\[
\sum_{j=1}^h a_j \alpha_j(H + \lambda H') = 0.
\]

But since \( \alpha_1(H + \lambda H'), \ldots, \alpha_v(H + \lambda H') \in \text{Im} \varphi_H \) it follows that \( a_{v+1} = \cdots = a_h = 0 \). Hence we have a nontrivial relation

\[
\sum_{j=1}^v a_j \alpha_j(H + \lambda H') = 0
\]

which is absurd.

Now take four linearly independent planes \( H_0 = H, H_1, H_2, H_3 \).
The fact that $\alpha H_i \in \Im \varphi_H$ ($i = 1, 2, 3$) means that, if $\alpha T$ is the image of $\alpha$ in $H^1(\mathcal{I}_T(t-1))$ and $L_i = H_i \cap H$, we have $\alpha_T L_i = 0$ in $H^1(\mathcal{I}_T(t))$.

Let $\sigma \in H^0(\mathcal{O}_C(t-1))$ be such that the image of $\sigma$ in $H^1(\mathcal{I}_C(t-1))$ is $\alpha$ and let $\sigma_T$ be the restriction of $\sigma$ to $T$. We have

$$\sigma_T L_i \in H^0(\mathcal{O}_H(t)).$$

We identify $H$ with $\mathbb{P}^2$ and let $R = \mathbb{K}[X_1, X_2, X_3]$ be the polynomial ring; we can suppose $L_i = X_i, i = 1, 2, 3$.

Then there are forms $Q_i \in R_t$ such that $\sigma_T X_i = Q_i$ in $H^0(\mathcal{O}_H(t))$. From this it follows that $Q_i X_j = Q_j X_i$ mod $H^0(\mathcal{I}_H(t+1))$, for all $i, j = 1, 2, 3$.

We use the following

**Lemma.** With the notation as before let $J \subset R$ be the homogeneous ideal of $\Gamma$ i.e. $J = \sum_{n=0}^{\infty} H^0(\mathcal{I}_H(n))$. Then the following are equivalent:

1. $\text{Tor}_1^R(J, R)_{t+2} = 0$,
2. for every triple $(Q_1, Q_2, Q_3), Q_i \in R_t, i = 1, 2, 3, \text{ such that } X_i Q_j - X_j Q_i \in J_{t+1}$

for all $i, j = 1, 2, 3$, there exists $Q \in R_{t-1}$ such that

$$Q_i - X_i Q \in J_t, \ i = 1, 2, 3.$$

**Proof.** See [L, Lemma, p. 141].

We observe that $\dim \text{Tor}_1^R(J, R)_{t+2}$ is the number of sinygies of $J$ in degree $t + 2$: this is easily seen by taking a free resolution of the $R$-module $J$ and tensoring by $\mathbb{K}$.

In our case, since $\Gamma$ is a c.i. $(s, t)$ and $s \geq 3$, $\text{Tor}_1^R(J, R)_{t+2} = 0$, hence by the lemma we have

$$\sigma_T X_i = Q X_i \in H^0(\mathcal{O}_H(t));$$

but we can assume that $X_i$ does not vanish on any point of $\Gamma$, hence $\sigma_T = Q$ in $H^0(\mathcal{O}_H(t-1))$.

This implies that the image of $\alpha$ is zero $H^1(\mathcal{I}_H(t-1))$, hence there exist $\beta \in H^1(\mathcal{I}_C(t-2))$ such that $\alpha = \beta H$.

So we have that, for a generic plane $H$, the map

$$\varphi_{H^2}: H^1(\mathcal{I}_C(t-2)) \rightarrow H^1(\mathcal{I}_C(t))$$

has a kernel of dimension 1, since the map $H^1(\mathcal{I}_C(t-2)) \rightarrow H^1(\mathcal{I}_C(t-1))$ is injective.

Let us continue, assuming by induction that the map

$$\varphi_{H^l}: H^1(\mathcal{I}_C(t-l)) \rightarrow H^1(\mathcal{I}_C(t))$$

has, for a generic $H$, a kernel of dimension 1 and prove that $\varphi_{H^{l+1}}$ has the same property. This proves the theorem since, for $l = t$, it is $H^1(\mathcal{I}_C) = 0$.

Suppose $\alpha H^l = 0$ and prove first that $\alpha H^l \in \Im \varphi_H$ for any $H'$. The proof is as before; let $\alpha_1, \ldots, \alpha_m = \alpha$ be a basis of $H^1(\mathcal{I}_C(t-l))$ and assume that $\alpha H^l \notin \Im \varphi_H$. We can assume that $\alpha_1 H^l, \ldots, \alpha_m H^l$ generate $V = \Im \varphi_H \cap \Im \alpha H^l$ and $\alpha_{m+1} H^l, \ldots, \alpha_m H^l$ are linearly independent and generate a subspace $S$ such that $S \cap V = (0)$. 

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Consider \((H + \lambda H')^l\); for \(\lambda\) generic, \(\varphi_{(H + \lambda H')^l}\) has rank \(m - 1\) and \(\alpha_1(H + \lambda H')^l, \ldots, \alpha_{m-1}(H + \lambda H')^l\) are linearly independent.

We have a relation
\[
\sum_{j=1}^{m} a_i \alpha_i(H + \lambda H')^l = 0
\]
but since \(\alpha_1(H + \lambda H')^l, \ldots, \alpha_{v}(H + \lambda H')^l \in \text{Im} \varphi_H\) it follows that \(a_{v+1} = \cdots = a_m = 0\).

Hence we have a nontrivial relation
\[
\sum_{j=1}^{v} a_i \alpha_i(H + \lambda H')^l = 0
\]
which is absurd.

Now we consider the planes \(H_0 = H, H_1, H_2, H_3\) and the plane \(H_1 + \lambda H_2 + \mu H_3\); we have \(\alpha(H_1 + \lambda H_2 + \mu H_3)^l \in \text{Im} \varphi_H\) for every \(\lambda, \mu\).

Developing the power we get that
\[
\alpha H_1, H_2 \cdots H_v \in \text{Im} \varphi_H
\]
for every \(1 \leq i_1 \leq i_2 \leq \cdots \leq i_v \leq 3\).

Using the lemma we get \(\alpha = \beta H\) with \(\beta \in H^1(\mathcal{J}(t - l - 1))\).

Case 2. \(d = s^2\) and \(C\) does not lie on a surface of degree \(s\). In this case the map
\[
\varphi_H: H^1(\mathcal{J}(s - 1)) \rightarrow H^1(\mathcal{J}(s))
\]
has, for a generic \(H\), a kernel of dimension 2.

Let \(\alpha, \tilde{\alpha}\) be a basis of \(\ker \varphi_H\). We prove that, for every \(H'\), \(\alpha H' \in \text{Im} \varphi_H\) and \(\tilde{\alpha} H' \in \text{Im} \varphi_H\) from which the theorem follows in a similar way as in case 1.

Let \(V = \text{Im} \varphi_H \cap \text{Im} \varphi_{H'}\) and let \(W\) be the subspace generated by \(\alpha H', \tilde{\alpha} H'\).

(a) Suppose first that \(V \cap W = (0)\).

Let \(\alpha_1, \ldots, \alpha_{h-1} = \alpha, \alpha_h = \tilde{\alpha}\) be a basis of \(H^1(\mathcal{J}(s - 1))\); we can assume that \(\alpha_1 H', \ldots, \alpha_v H' \in V\) and \(\alpha_{v+1} H', \ldots, \alpha_h H'\) are linearly independent and generate a subspace \(S\) such that \(S \cap V = (0)\).

We obtain a relation, for a generic \(\lambda\),
\[
\sum_{j=1}^{v} a_i \alpha_i(H + \lambda H') = 0
\]
which is absurd.

(b) Suppose \(\dim V \cap W = 1\). We can assume \(\alpha H' \in V, \tilde{\alpha} H' \notin V\).

We can choose a basis \(\alpha = \alpha_1, \ldots, \alpha_h = \tilde{\alpha}\) of \(H^1(\mathcal{J}(s - 1))\) such that \(\alpha_1 H', \ldots, \alpha_v H' \in V\) but \(\alpha_{v+1} H', \ldots, \alpha_h H'\) are linearly independent and generate a subspace \(S\) such that \(S \cap V = (0)\).

We have two relations
\[
\sum_{j=1}^{v} a_i \alpha_i(H + \lambda H') = 0, \quad \sum_{j=1}^{v} b_i \alpha_i(H + \lambda H') = 0
\]
from which a nontrivial relation \(\sum_{i=2}^{v} c_i \alpha_i(H + \lambda H') = 0\) which is absurd.
REMARK 7. Theorem 6 holds also in the case when $\kappa$ has characteristic $p > 0$, provided $p$ is sufficiently high; precisely $p$ must be greater than the coefficients involved in the power $(H_1 + \lambda H_2 + \mu H_3)^l$, $1 \leq l \leq t - 1$.

REMARK 8. With the same proof of Theorem 6, case 1 (with $s$ instead of $t$) one can prove that, if $\Gamma$ is a c.i. $(s, t)$, $t \geq s + 1$, then $C$ lies on a surface of degree $s$. In this way one can avoid the use of Laudal's theorem.

REMARK 9. The same idea of the proof of Theorem 6 can be used to give an alternative and simpler proof of Laudal's Theorem, avoiding the use of deformation theory (see [S] for details).

REFERENCES


DIPARTIMENTO DI MATEMATICA, UNIVERSITÁ DI CATANIA, VIALE A. DORIA, 6, 95125 CATANIA, ITALY