INVENTARIES OF FINITE ABELIAN GROUPS ACTING ON THE ALGEBRA OF TWO 2 x 2 GENERIC MATRICES

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ABSTRACT. In this paper, we discuss the finite generation problem for the invariant subalgebras of finite abelian groups which act linearly on the 2 x 2 generic matrix algebra, and we obtain some conditions on the groups to ensure that their invariant subalgebras are finitely generated.

Introduction. Let \( R_{(m)} = K\{X, Y\} \) be a generic matrix algebra generated by two \( m \times m \) generic matrices over a field \( K \). A theorem of Fisher and Montgomery [2] shows that if \( G = \langle g \rangle \) is a finite cyclic group such that \( \text{char}\ K \nmid |G| \), then \( R_{(m)}^G \), the subalgebra of invariants, is not finitely generated whenever \( g \) is not scalar and \( m \geq |G| - \lfloor \sqrt{|G|} \rfloor + 1 \).

When \( m = 2 \) and \( G \) is a finite subgroup of \( SL(2, K) \), Formanek and Schofield [3] have proved that if \( \text{char}\ K \nmid |G| \), then \( R_{(2)}^G \) is always finitely generated.

In this paper we consider the case when \( m = 2 \) and \( G \) is a finite abelian subgroup of \( GL(2, K) \) such that \( \text{char}\ K \mid |G| \). In this case we may assume that \( K \) is algebraically closed. So \( G \) acts diagonally on \( R = R_{(2)} \) and \( R^G \) is generated by some monomials in the generators of \( R \).

We obtain necessary and sufficient conditions on \( G \) to ensure that \( R^G \) is finitely generated. They are as follows: \( R^G \) is finitely generated if and only if \( G \) contains no pseudoreflections or, equivalently, \( G = \langle g \rangle \) is cyclic and the eigenvalues of \( g \) in some extension field of \( K \) have the same order. In some sense, this provides a converse to Theorem 1 of Fisher and Montgomery [2].

1. For cyclic groups.

LEMMA 1.1 (FORMANEK AND SCHOFIELD). Let \( G \) be any finite subgroup of \( GL(2, K) \) such that \( \text{char}\ K \mid |G| \) and let \( [R, R] = J \) be the commutator ideal of \( R \). Then

(i) for each \( m \geq 1 \), \( J^m = J(XY - YX)^{m-1} \), and
(ii) \( G \) acts linearly on \( R/J^2 \) and \( (R/J^2)^G \) is a finitely generated \( K \)-algebra.

Proof. See [3, Lemmas 6 and 7].

LEMMA 1.2. Let \( u \in R \) be homogeneous and \( Z = XY - YX \). Then

\[
    u = vZ + aY^T X^8 + \sum b_{i,j,k,l} Y^i Z Y^j X^k X^l
\]

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where \( v \in R \) is homogeneous, \( r, s, i, j, k, l \) are nonnegative integers, and \( a, b, i, j, k, l \) are in \( K \). Moreover, \( u, v Z, Y^r X^s \) and \( Y^i X^j Y^k X^l \) all have the same degree in \( X \) and the same degree in \( Y \).

**Proof.** Since \( u \) is homogeneous, \( u \) is a sum of a finite number of monomials in \( X \) and \( Y \)—all of which have the same degree in \( X \) and the same degree in \( Y \). Consider one of these monomials \( M \). If for each \( X \) in \( M \) no \( Y \) occurs immediately adjacent to \( X \) on the right, then \( M \) has the form \( a Y^r X^s \). If \( Y \) occurs adjacent to \( X \) on the right, then replace \( "XY" \) by \( "Z + YX" \) to get \( M \) with \( "XY" \) replaced by \( "YX" \) plus a new term involving \( Z \). Note that this transformation preserves homogeneity. By iterating this process we reduce the original monomial \( M \) to one of the form \( a Y^r X^s \) at the expense of introducing some new terms of the form \( \frac{N}{b} = bu_1 u_2 \ldots u_t \) where \( b \in K \), one \( u_\lambda \) is \( Z \) and the rest are \( X \) or \( Y \).

If for each \( X \) in \( N \) no \( Y \) occurs immediately adjacent to \( X \) on the right, then \( N \) has the form \( b_i, j, k, l Y^i X^j Y^k X^l \). If \( Y \) occurs adjacent to \( X \) on the right, then replace \( "XY" \) by \( "Z + YX" \) to get \( N \) with \( "XY" \) replaced by \( "YX" \) plus a new term involving two factors of \( Z \). By iterating this process the original term \( N \) is reduced to one of the form \( b_i, j, k, l Y^i X^j Y^k X^l \) at the expense of introducing some new terms of the form \( p = cu_1, u_2, \ldots, u_t \) where \( c \in K \) and there exist \( \lambda \) and \( \rho \) with \( 1 \leq \lambda < \rho \leq t \) such that \( u_\lambda = Z = u_\rho \). However, each of these new terms \( P \) is in \( J^2 \). By Lemma 1.1, \( J^2 = JZ \) so \( P = vZ \) for some \( v \in J \). Since \( P \) is homogeneous and \( P = vZ \), we have that \( v \) is homogeneous.

It is clear from the proof that all monomials in \( u \) and \( vZ, Y^r X^s, Y^i X^j Y^k X^l \) have the same degree in \( X \) and have the same degree in \( Y \), i.e., \( X\text{-deg} = s = j + l \) and \( Y\text{-deg} = r = i + \kappa \).

In order to prove the main theorem of this section (Theorem 1.7), we need to define some integer-valued mappings. Fix integers \( n \) and \( t \) such that \( 2 \leq n \), \( 0 \leq t < n \), and \( (n, t) = 1 \).

(1) for each nonnegative integer \( p \), let \( \alpha(p) \) be the smallest nonnegative integer such that \( p(t + 1) + \alpha(p) \equiv 0 \pmod{n} \), i.e., \( \alpha(p) = \min\{q \mid q \geq 0, p(t + 1) + q \equiv 0 \pmod{n}\} \).

(2) For each pair of nonnegative integers \( p \) and \( q \), let \( \beta(p, q) \) be the smallest nonnegative integer such that \( p(t + 1) + q + \beta(p, q)t \equiv 0 \pmod{n} \), i.e.,

\[
\beta(p, q) = \min\{r \mid r \geq 0, p(t + 1) + q + rt \equiv 0 \pmod{n}\}.
\]

(3) For each triple of nonnegative integers \( p, q, \) and \( r \), let \( \gamma(p, q, r) \) be the smallest nonnegative integer such that \( p(t + 1) + q + rt + \gamma(p, q, r) \equiv 0 \pmod{n} \).

(4) For each quadruple of nonnegative integers \( p, q, r, \) and \( s \), let \( \delta(p, q, r, s) \) be the smallest nonnegative integer such that \( p(t + 1) + q + rt + s + \delta(p, q, r, s)t \equiv 0 \pmod{n} \).

Since \( (n, t) = 1 \), the values \( \alpha(p), \beta(p, q), \gamma(p, q, r), \) and \( \delta(p, q, r, s) \) exist uniquely for all nonnegative integers \( p, q, r, \) and \( s \). Moreover, all these values are less than \( n \), i.e., \( 0 \leq \alpha, \beta, \gamma, \delta < n \).

Now we define some special subsets of \( R \) by using the mappings \( \alpha, \beta, \gamma, \) and \( \delta \) defined above.
DEFINITION.

\[ B_1 = \{ X^{\alpha(p)} Z^p \mid 1 \leq p \leq n \}, \]
\[ B_2 = \{ Y^{\beta(p,q)} X^q Z^p \mid 1 \leq p \leq n, 0 \leq q \leq n - 1 \}, \]
\[ B_3 = \{ ZY^{\beta(p,q)} X^q Z^p-1 \mid 1 \leq p \leq n, 0 \leq q \leq n - 1 \}, \]
\[ B_4 = \{ X^{\gamma(p,q,r)} ZY^{r} X^q Z^p-1 \mid 1 \leq p \leq n, 0 \leq q, r \leq n - 1 \}, \]
\[ B_5 = \{ Y^{\delta(p,q,r,s)} X^r ZY^{s} X^q Z^p-1 \mid 1 \leq p \leq n, 0 \leq q, r, s \leq n - 1 \}, \]
\[ B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5. \]

REMARKS 1.3. (i) Since \( B_1, B_2, B_3, B_4, \) and \( B_5 \) are finite subsets of \( R, \) \( B \) is also a finite subset of \( R. \)

(ii) Let \( n \) be a positive integer such that \( \text{char } K \nmid n \) and let \( \omega \) be a primitive \( n^\text{th} \) root of unity. If \( t \) is a positive integer such that \( (n,t) = 1, \) then we define an automorphism \( g \) on \( R \) by setting \( X^g = \omega X \) and \( Y^g = \omega^t Y. \) By the definition of the mappings \( \alpha, \beta, \gamma, \) and \( \delta, \) we have that \( B \) is a subset of \( RG \) where \( G = \langle g \rangle. \)

(iii) If \( G \) is as in (ii) and \( 0 \neq u \in RG \) is homogeneous, then each monomial in \( u \) lies in \( RG. \)

LEMMA 1.4. Let \( G = \langle g \rangle \) be a cyclic group of order \( n \) acting on \( R \) by \( X^g = \omega X \) and \( Y^g = \omega^t Y \) where \( \omega \) is a primitive \( n^\text{th} \) root of unity such that \( \text{char } K \nmid n \) and \( (t,n) = 1. \) Let \( p \) be a positive integer.

(i) If \( Y^i X^j Z^p \in RG \) for some nonnegative integers \( i \) and \( j, \) then there exist \( w \) in \( RG \) and \( b \) in \( B \) such that \( Y^i X^j Z^p = wb. \)

(ii) If \( Y^i X^j ZY^{\kappa} X^l Z^p \in RG \) for some nonnegative integers \( i, j, \kappa, l \) then there exist \( w \) in \( RG \) and \( b \) in \( B \) such that \( Y^i X^j ZY^{\kappa} X^l Z^p = wb. \)

PROOF. The proof of (i) is similar to that of (ii), so we will prove (ii). If \( p \geq n, \) then \( Y^i X^j ZY^{\kappa} X^l Z^p = (Y^i X^j ZY^{\kappa} X^l Z^{p-n}) Z^n. \) Let \( b = Z^n \) and \( w = Y^i X^j ZY^{\kappa} X^l Z^{p-n}. \) Then \( b \in B_1 \subset B \subset RG. \) Since \( R \) has no zero divisors, \( w \) must be in \( RG. \)

Now assume that \( p < n. \) For convenience let \( u = Y^i X^j ZY^{\kappa} X^l Z^p. \)

(1) If \( l \geq \alpha(p) = \alpha, \) then let \( b = X^{\alpha(p)} Z^p \) and \( w = Y^i X^j ZY^{\kappa} X^{l-\alpha}. \) Then \( u = wb \) and \( b \in B_1 \subset B. \) Again \( w \in RG. \)

(2) If \( l < \alpha(p) \) and \( \kappa \geq \beta(p,l) = \beta, \) then let \( b = Y^{\beta(p,l)} X^l Z^p \) and \( w = Y^i X^j ZY^{\kappa-\beta}. \) Then \( b \in B_2 \subset B, \) \( u = wb, \) and \( w \in RG. \)

(3) If \( l < \alpha(p), \kappa < \beta(p,l), \) and \( \kappa = \beta(p+1,l), \) then let \( b = ZY^{\kappa} X^l Z^p \) and \( w = Y^i X^j. \) Then \( b \in B_3 \subset B, \) \( u = wb \) and \( w \in RG. \)

(4) If \( l < \alpha(p), \kappa < \alpha(p,l), \kappa \neq \beta(p+1,l), \) and \( j \geq \gamma(p+1,l,\kappa) = \gamma \) then let \( b = X^{\gamma} ZY^{\kappa} X^l Z^p, \) \( w = Y^i X^{j-\gamma}. \) Then \( b \in B_4 \subset B, \) \( u = wb \) and \( w \in RG. \)

(5) If \( l < \alpha(p), \kappa < \beta(p,l), \kappa \neq \beta(p+1,l), j < \gamma(p+1,l,\kappa), \) then \( i \geq \delta = \delta(p+1,l,\kappa,j). \) For otherwise \( u \notin RG. \) So let \( w = Y^{i-\delta} \) and \( b = Y^i X^{j} ZY^{\kappa} X^l Z^p. \) Then \( u = wb, b \in B_5 \subset B \) and \( w \in RG. \) The lemma is proved.

PROPOSITION 1.5. Let \( G = \langle g \rangle \) be a cyclic group of order \( n \) acting on \( R \) by \( X^g = \omega X \) and \( Y^g = \omega^t Y \) where \( \omega \) is a primitive \( n^\text{th} \) root of unity such that \( \text{char } K \nmid n \) and \( (t,n) = 1. \) If \( u \in R \) is homogeneous and \( uZ^p \in RG \) (\( p \geq 1 \)), then \( uZ^p = \sum w_ub_u \) is a finite sum where \( w_u \in RG \) and \( b_u \in B. \) Moreover \( uZ^p \) and \( w_ub_u \) have the same degree in \( X \) and the same degree in \( Y. \)
PROOF. The proof is by reverse induction on $p$. If $p \geq n$ then $Z^n \in B_1 \subset B$ and $uZ^p = (uZ^{p-n})Z^n$. Let $w = uZ^{p-n}$ and $b = Z^n$. Then $uZ^p = wb$ and $w \in R^G$. Assume $1 \leq p < n$. By Lemma 1.2 we have that

\[ uZ^p = vZ^{p+1} + aY^sZ^p + \sum b_{j,k,l}Y^jX^kZ^l, \]

where $v \in R$ is homogeneous, $a, b_{j,k,l} \in K$, and $uZ^p$, $vZ^{p+1}$, $Y^sZ^p$ and $Y^jX^kZ^l$ all have the same degree in $X$ and the same degree in $Y$. By Remark 1.3(iii), each term on the right-hand side of ($\ast$) lies in $R^G$. From the homogeneity of $v$ and the reverse induction hypothesis $vZ^{p+1} = \sum w_v b_v$ for some $w_v \in R^G$ and $b_v \in B$. The other terms on the right-hand side of ($\ast$) are also of the form $wb$, $w \in R^G$, $b \in B$ by Lemma 1.4. Now the proof is completed.

THEOREM 1.6 [2, THEOREM 1]. Let $G = \langle g \rangle$ be a cyclic group acting on $R$ by $x^g = \omega x$ and $y^g = \lambda y$ where $\omega$ and $\lambda$ are $n$th roots of unity. If $\omega$ and $\lambda$ have distinct orders, then $R^G$ is not finitely generated.

We prove the converse of the above theorem. The proof is similar to that of [3, Theorem 8].

THEOREM 1.7. Let $G = \langle g \rangle$ be a cyclic group of order $n$ acting on $R$ by $x^g = \omega x$ and $y^g = \lambda y$ where both $\omega$ and $\lambda$ have the same (multiplicative) order $n$. Then $R^G$ is finitely generated.

PROOF. We may assume that $\lambda = \omega^t$ where $(n,t) = 1$. Since char $K \nmid |G|$, the natural homomorphism $\theta : R^G \to (R/J^2)^G$ is surjective. By Lemma 1.1(i) there exist $v_1, v_2, \ldots, v_m$ in $R^G$ whose images generate $(R/J^2)^G$ as a $K$-algebra. As we stated before, we may assume that $v_1, v_2, \ldots, v_m$ are monomials since $G$ consists of diagonal matrices. Let $A = \{v_1, v_2, \ldots, v_m\}$ and let $T$ be the subalgebra of $R^G$ which is generated by $A \cup B$ where $B$ is as in the definition above. Then $T$ is finitely generated and it suffices to prove that $R^G = T$. It is obvious that $T \subset R^G$. Let $u$ be any homogeneous element of $R^G$. We will show that $u \in T$ by induction on the degree of $u$. If $\deg u = 0$, then $u \in K \subset T$. Assume that $\deg u \geq 1$. Since $A = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m\}$, where $\bar{v}$ denotes the image of $v \in R^G$ under $\theta$, generates $(R/J^2)^G$, there exists a polynomial $f = f(x_1, x_2, \ldots, x_m)$ in the free algebra $K(x_1, x_2, \ldots, x_m)$ such that $\bar{u} = f(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m)$. So $u - f(v_1, v_2, \ldots, v_m) \in \text{Ker} \theta = R^G \cap J^2 \subset J^2$. By Lemma 1.1(i), $u - f(v_1, v_2, \ldots, v_m) = vZ$ for some $v \in J$. Moreover $vZ = u - f(v_1, v_2, \ldots, v_m) \in R^G$. Since each $v_i$ is a monomial and $J^2$ is homogeneous, we may assume that $v$ is homogeneous and $\deg vZ \leq \deg u$. So by Proposition 1.5, $vZ = \sum w_v b_v$ where $w_v \in R^G$ and $b_v \in B$. But $\deg w_v < \deg w_v + \deg b_v = \deg w_v b_v = \deg vZ \leq \deg u$ for each $v$, so by the induction hypothesis, $w_v \in T$. Hence $vZ = \sum w_v b_v \in T$. Thus $u = vZ + f(v_1, v_2, \ldots, v_m) \in T$ and so $R^G = T$ is finitely generated.

DEFINITION A diagonalizable $m \times m$ matrix $g$ over $K$ is said to be a pseudo-reflection if $\text{rank}(g - I_m) = 1$, where $I_m$ is the $m \times m$ identity matrix or, equivalently, if 1 is an eigenvalue of $g$ of multiplicity exactly $m - 1$.

If $g \in GL(2, K)$ has eigenvalues $\omega_1$ and $\omega_2$ which are roots of unity of distinct orders, then for some $n \geq 1$, $g^n$ is a pseudoreflection. Hence for a finite subgroup $G$ of $GL(2, K)$ such that char $K \nmid |G|$, the following two statements are equivalent: (i) $G$ has an element $g$ whose eigenvalues have distinct orders, and (ii) $G$ contains a pseudoreflection.
COROLLARY 1.8. Let $G$ be a cyclic subgroup of $GL(2,K)$ of order $n$ with \text{char} \, $K$ $\nmid$ $n$. Then $R^G$ is finitely generated if and only if $G$ contains no pseudo-reflections.

PROOF. Let $g$ be the generator of $G$. Since we may assume that $K$ is algebraically closed, $g$ is diagonalizable.

If $R^G$ is finitely generated, then the eigenvalues of $g$ have the same order by Theorem 1.6. So $G = \langle g \rangle$ cannot contain pseudo-reflections.

Conversely, if $G$ contains no pseudo-reflections, then the eigenvalues of $g$ must have the same order. So $R^G$ is finitely generated by Theorem 1.7.

2. For abelian groups. In this section, we let $G$ be a finite abelian subgroup of $GL(2,K)$ such that $2 \leq |G|$ and \text{char} \, $K$ $\nmid$ $|G|$. Since we may assume that $K$ is algebraically closed, $G$ can be considered as a group of diagonal matrices.

First we need the following result of Fisher and Montgomery [2, Proposition 1].

PROPOSITION 2.1. Let $n$ and $N$ be positive integers and let $f(x,y)$ be a nonzero homogeneous polynomial in $K(x,y)$ of $x$-degree $n + 1$ and $y$-degree $N$ of the form $f(x,y) = x^ny^N + x + yf_1(x,y) + f_2(x,y)y$ for some $f_i \in K(x,y)$. Then $f(x,y)$ is not an identity for $m \times m$ matrices for any $m \geq 2$.

LEMMA 2.2. Let $G$ be a finite abelian subgroup of $GL(2, \mathbb{A})$ such that \text{char} \, $\mathbb{A}$ $\nmid$ $|G|$. If $R^G$ is finitely generated then $XY^q \subseteq R^G$ and $X^pY \subseteq R^G$ for some $p \geq 1$ and $q \geq 1$.

PROOF. Suppose $R^G$ is finitely generated. By symmetry it suffices to prove $XY^q \subseteq R^G$ for some $q \geq 1$. Let $m$ be the smallest positive integer such that $X^mY^q \subseteq R^G$ for some $q \geq 1$; such an $m$ exists since $X^nY^m \subseteq R^G$ where $|G| = n$. Hence $X^mY^q+n\kappa \subseteq R^G$ for each $\kappa \geq 0$. Assume $m \geq 2$. Note if $u \in R^G$ and $v \in R$ are monomials such that $u$ and $v$ have the same degree in $X$ and the same degree in $Y$, then $v \in R^G$ since $G$ is diagonalizable. Hence $X^{m-1}Y^q+n\kappa \subseteq R^G$ for each $\kappa \geq 0$. Since $m - 1 \geq 1$, $X^{m-1}Y^q+n\kappa \subseteq R^G$ has no proper initial segments which lie in $R^G$ by definition of $m$. Also note that since $G$ is diagonal, if $u \in R^G$ is homogeneous, then each monomial of $u$ lies in $R^G$. So we may choose a set of generators $\{u_1, u_2, \ldots, u_r\}$ of $R^G$, all of which are monomials in $X$ and $Y$. Let $\kappa$ be an integer greater than the maximal of the degrees of $u_i$’s. Since $X^{m-1}Y^q+n\kappa \subseteq R^G$, it is a finite sum of monomials in $u_1, u_2, \ldots, u_r$, that is $X^{m-1}Y^q+n\kappa \subseteq R^G$ is homogeneous, we may assume that each $a_iu_1, u_2, \ldots, u_s$ has $X$-degree $m$ and $Y$-degree $q+\kappa$. Consider any monomial $a_iu_1, u_2, \ldots, u_s$ appearing in this expression of $X^{m-1}Y^q+n\kappa \subseteq R^G$. Then $s > n \geq 2$, since $\kappa n < m+q+\kappa = \text{deg}(X^{m-1}Y^q+n\kappa \subseteq R^G) = \text{deg}(a_iu_1, u_2, \ldots, u_s) < \kappa s$. Since the $X$-degree of $a_iu_1, u_2, \ldots, u_s$ is $m$ and $u_{ij} \in R^G$ for each $j = 1, 2, \ldots, s$, from the definition of $m$, exactly one $u_{ij}$ has $X$ as its factor. Thus each monomial $a_iu_1, u_2, \ldots, u_s$ has the form either $vY$ or $Yv$ for some $v$ in $R$. But this is impossible by Proposition 2.1. Hence $m = 1$.

Actually the converse of Lemma 2.2 is true.

THEOREM 2.3. Let $G$ be a finite abelian subgroup of $GL(2,K)$ such that \text{char} \, $K$ $\nmid |G|$. Then the following are equivalent:

(i) $R^G$ is finitely generated,
(ii) \( G \) contains no pseudoreflections,
(iii) \( G = \langle g \rangle \) is cyclic and the eigenvalues of \( g \) in some extension field of \( K \) have the same order.

**Proof.** (iii) implies (i) is just Theorem 1.7.

For (i) implies (ii), assume that \( G \) contains a pseudoreflection \( g \). By symmetry we may assume \( g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \), for some \( \alpha \neq 1 \). Then for any \( q \geq 1 \), \( (XY^q)^g = \omega XY^q \neq XY^q \). Since \( XY^q \notin R^G \) for any \( q \geq 1 \), \( R^G \) is not finitely generated by Lemma 2.2.

For (ii) implies (iii), suppose that \( G \) contains no pseudoreflections. If \( G \) is not cyclic, then \( G = G_1 \oplus \cdots \oplus G_r \) where each \( G_i = \langle g_i \rangle \) is cyclic of order \( n_i \) with \( n_1 | n_2 | \cdots | n_r \) and \( r \geq 2 \). Hence for some primitive \( n_r^{th} \) root of unity \( \omega \), the generators \( g_1 \) and \( g_r \) have the following forms:

\[
g_1 = \begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix} \quad \text{and} \quad g_r = \begin{pmatrix} \omega & 0 \\ 0 & \omega^t \end{pmatrix},
\]

where \( at \neq b \) (mod \( n_r \)) since \( g_1 \notin \langle g_r \rangle \). But

\[
g_1 g_r^{-a} = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{b-\alpha t} \end{pmatrix}
\]

is a pseudoreflection since \( \omega^{b-\alpha t} \neq 1 \). This gives us the desired contradiction. So \( G \) is cyclic. Since \( G \) contains no pseudoreflections, the eigenvalues of the generator \( g \) of \( G \) have the same order.

**Corollary 2.4.** Let \( G \) be as in the theorem and suppose that \( |G| \) is not a prime number. Then \( R^G \) is finitely generated if and only if \( R^H \) is finitely generated for each proper subgroup \( H \) of \( G \).

**Proof.** If \( R^G \) is finitely generated, then by the theorem, \( G \) (and each subgroup \( H \)) contains no pseudoreflections. Thus using the theorem again, \( R^H \) is finitely generated.

Conversely if \( R^G \) is not finitely generated, then \( G \) contains a pseudoreflection \( g \). We may assume that \( g \) has prime order; if not, replace \( g \) by some power \( g^t \). But then \( H = \langle g \rangle \) has prime order, and so \( H \neq G \) since \( |G| \) is not prime. Thus \( R^H \) is not finitely generated, for some proper subgroup \( H \) of \( G \).

The following corollary was noticed by R. Guralnick.

**Corollary 2.5.** Let \( H \) be a finite subgroup of \( GL(2, \mathbb{K}) \) of odd order such that \( \text{char} \mathbb{K} \nmid |H| \). Then \( R^H \) is finitely generated if and only if \( H \) contains no pseudoreflections.

**Proof.** As we stated before, we may assume that \( K \) is algebraically closed. So, if \( n \) is the degree of an irreducible representation, then \( n \) must divide \( |H| \) by [1, Theorem 53.17]. By Theorem 2.3, it is enough to show that \( H \) is abelian. Let \( V \) be a 2-dimensional vector space over \( K \). Then \( H \) is a group of linear operators of \( V \). If \( n \) is the degree of an irreducible representation of \( H \), then \( n \) is odd since \( n \mid |H| \).

Hence the \( K \)-dimension of any irreducible \( K[H] \)-submodule of \( V \) is odd and equal to or less than 2 = \( \dim V \), so it is 1. Thus \( H \) is diagonalizable, so \( H \) is abelian.

**Note.** A natural question is whether our theorem can be extended to finite solvable groups. An obstacle to extending it to finite solvable groups—by the
obvious approach—is a theorem of R. M. Guralnick [4] which states that \( R^G \) cannot be isomorphic to a generic matrix algebra.

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**REFERENCES**