INVARIANT SUBSPACES FOR ALGEBRAS
OF SUBNORMAL OPERATORS. II

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ABSTRACT. We continue our study of hyperinvariant subspaces for rationally cyclic subnormal operators. We establish a connection between hyperinvariant subspaces and weak-star continuous point evaluations on the commutant.

Introduction. Let $K$ be a compact subset of the complex plane $\mathbb{C}$ and let $R(K)$ denote the algebra of rational functions with poles off $K$. For a positive measure $\mu$ with support in $K$ let $R^2(K,\mu)$ denote the closure of $R(K)$ in $L^2(\mu)$. Every rationally cyclic subnormal operator is unitarily equivalent to multiplication by $z$, $M_z$, on an $R^2(K,\mu)$-space [1, p. 146]. Under this representation each operator that commutes with $M_z$ is represented by multiplication by a function in $R^2(K,\mu) \cap L^\infty(\mu)$, and conversely [1, p. 147].

In [2] we proved the existence of invariant subspaces for the algebra $R^2(K,\mu) \cap L^\infty(\mu)$. In this paper we show there exist weak-star continuous point evaluations on $R^2(K,\mu) \cap L^\infty(\mu)$ if $L^2(\mu) \neq R^2(K,\mu)$, and we show that these point evaluations give rise to the kinds of invariant subspaces found in [2]. Finally, we establish a connection between analytic bounded point evaluations on $R^2$-spaces and $R^p$-spaces for $2 < p < 4$.

Notation. Let $m$ denote area measure on $\mathbb{C}$. For a complex measure $\nu$ with compact support in $\mathbb{C}$ let $\nu(z) = \int (\xi - z)^{-1} d\nu(\xi)$ and $\nu(z) = \int |\xi - z|^{-1} d|\nu|(\xi)$. By Tonelli's theorem $\nu(z) < \infty$ $m$-a.e. and hence the Cauchy transform $\tilde{\nu}$ is defined $m$-a.e. For $g$ in $L^1(|\nu|)$ let $\tilde{g} = (g d\nu)^\ast$.

THEOREM. Let $g \in R^2(K,\mu) \cap L^\infty(\mu)$ and let $A = \{z: g(z) \neq 0\}$. Then for $m$-a.e. $z$ in $A$ there exists a weak-star continuous multiplicative linear functional $e_z$ on $R^2(K,\mu) \cap L^\infty(\mu)$ such that $e_z(f) = f(z)$ for each $f$ in $R(K)$. Moreover, for $m$-a.e. such $z$ there exist $x$ and $y$ in $R^2(K,\mu)$ such that $e_z(f) = \langle fx, y \rangle$ for each $f$ in $R^2(K,\mu) \cap L^\infty(\mu)$.

REMARK. Let $z$ in $A$ be such that both conclusions of the theorem hold, and let $x$ and $y$ be as in the second conclusion. Let $H$ be the closed linear span of $\{(\xi - z)fx: f \in R^2(K,\mu) \cap L^\infty(\mu)\}$ in $L^2(\mu)$. Since $y \perp H$, it follows that $H$ is a nontrivial hyperinvariant subspace for $M_z$ on $R^2(K,\mu)$.

LEMMA 1. Let $p \in (2,4)$ and let $s = 2/(p - 2)$. Suppose $f_n \in L^2(\mu)$ and $\|f_n\|_2 < 2^{-n}$ for each positive integer $n$. Then there exists a function $w: \mathbb{C} \rightarrow (0,1]$ such that $w^{-1} \in L^s(\mu)$, $f_n \in L^p(w d\mu)$ for each $n$, and $f_n \rightarrow 0$ in $L^p(w d\mu)$.

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756
PROOF. Let \( w = (1 + \sum_{n=1}^{\infty} |f_n|)^{2-p} \). Redefine \( w \) to be one on the set where the infinite series diverges. Since \( w^{-s} = (1 + \sum |f_n|)^2 \), it follows that \( w^{-s} \in L^1(\mu) \) and hence \( w^{-1} \in L^s(\mu) \). Finally, \( |f_n|^p w \leq |f_n|^2 \) implies that \( f_n \in L^p(w \, d\mu) \) for each \( n \) and that \( f_n \rightarrow 0 \) in \( L^p(w \, d\mu) \).

**Lemma 2.** Let \( p \in (2,4) \), \( s = 2/(p - 2) \), and \( q = p(p - 1)^{-1} \). Suppose \( R^2(K,\mu) \neq L^2(\mu) \). Let \( f_1, f_2 \in R^2(K,\mu) \). Then there exists a function \( w : C \rightarrow (0,1] \) such that \( w^{-1} \in L^s(\mu) \), \( R^p(K, w \, d\mu) \neq L^p(w \, d\mu) \), and \( f_1, f_2 \in R^p(K, w \, d\mu) \).

**PROOF.** Define \( f_n \) to be \( f_1 \) when \( n \) is odd and \( f_2 \) when \( n \) is even. Let \( \{r_n\} \) be a sequence of functions from \( R(K) \) such that \( ||r_n - f_n||_2 < 2^{-n} \) for each \( n \). Applying Lemma 1 to the sequence \( \{r_n - f_n\} \), we obtain a function \( w : C \rightarrow (0,1] \) such that \( w^{-1} \in L^s(\mu) \) and \( r_n - f_n \rightarrow 0 \) in \( L^p(w \, d\mu) \). It follows immediately that \( f_1 \) and \( f_2 \) are in \( R^p(K, w \, d\mu) \). If \( g \in R^2(K,\mu)^{-1} \) then \( (g/w) \in L^q(w \, d\mu) \) by Hölder's inequality. Since \( R^2(K,\mu) \neq L^2(\mu) \), it follows that \( R^p(K, w \, d\mu) \neq L^p(w \, d\mu) \).

**Proof of the Theorem.** Fix \( p \) and \( q \) as in Lemma 2. Let \( B \) be a compact subset of \( A \) such that the following hold:

(i) \( (g \, d\mu)^+ \) is continuous;

(ii) \( (\zeta - z)^{-1} g(\zeta) \in L^q(\mu) \) for each \( z \in B \).

As a consequence of (i), \( \hat{g} | \overline{B} \) is continuous and hence \( \hat{g} \) is bounded away from zero on \( B \). Let \( E \) be the set of points in \( B \) that are of full density with respect to \( m \). We shall show that both conclusions of the theorem hold for each \( z \) in \( E \). We leave it as an exercise that \( m \)-a.e. \( z \) in \( A \) belongs to such an \( E \).

For \( z \in E \) let \( e_z(f) = \hat{g}(z)^{-1} \int (\zeta - z)^{-1} f(\zeta) g(\zeta) \, d\mu(\zeta) \) for each \( f \in R^2(K,\mu) \cap L^\infty(\mu) \). It is easy to check that \( e_z(f) = f(z) \) if \( f \in R(K) \). By the main proof in [2] there exist \( x \in R^2(K,\mu) \) and \( y \in L^2(\mu) \) such that \( e_z(f) = \langle f, y \rangle \) for each \( f \) in \( R^2(K,\mu) \cap L^\infty(\mu) \). Projecting \( y \) into \( R^2(K,\mu) \), we may assume that \( y \in R^2(K,\mu) \). It remains to show that \( e_z \) is multiplicative on \( R^2(K,\mu) \cap L^\infty(\mu) \).

For \( f \in R^2(K,\mu) \cap L^\infty(\mu) \) the map \( z \rightarrow e_z(f) \) is continuous on \( E \). This is a consequence of assumption (i). Thus it suffices to show that for \( f_1, f_2 \in R^2(K,\mu) \cap L^\infty(\mu) \) there exists a dense subset \( F \) of \( E \) such that \( e_z(f_1 f_2) = e_z(f_1) e_z(f_2) \) for each \( z \in F \). Let \( F \) be the set of \( z \in E \) for which

\[
\int |\zeta - z|^{-q}|(g/w)|^q w \, d\mu < \infty.
\]

By Tonelli's theorem \( m(E \setminus F) = 0 \). For \( z \in F \) the map \( f \rightarrow e_z(f) \) is \( L^p(w \, d\mu) \)-continuous on \( R(K) \). Since \( e_z \) is multiplicative on \( R(K) \), it follows from taking \( L^p(w \, d\mu) \) limits that \( e_z(f_1 f_2) = e_z(f_1) e_z(f_2) \) for each \( f \in R(K) \). Taking \( L^p(w \, d\mu) \) limits limits again, we obtain \( e_z(f_1 f_2) = e_z(f_1) e_z(f_2) \). That does it.

**Remark.** Let \( P^r(\mu) \) denote the closure of the polynomials in \( L^r(\mu) \). Suppose \( P^2(\mu) \) has no analytic bounded point evaluations. Then there exists a function \( f \) in \( P^2(\mu) \) that is essentially unbounded on every relatively open subset of the support of \( \mu \). If \( P^2(\mu) \neq L^2(\mu) \) then by Lemma 2 \( f \) is in a nontrivial \( P^r \)-space for every \( r \in (2,4) \). It follows that none of those \( P^r \)-spaces has any analytic bounded point evaluations. Thus, to show that every nontrivial \( P^2 \)-space has some analytic bounded point evaluations, it suffices to show that every nontrivial \( P^r \)-space has analytic bounded point evaluations for some \( r \in (2,4) \). It is true that bounded point evaluations exist for every nontrivial \( P^r \)-space for \( r > 2 \).
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