PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 3, November 1988

ON THE SPECTRAL PICTURE OF AN IRREDUCIBLE SUBNORMAL OPERATOR

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(Communicated by John B. Conway)

ABSTRACT. This paper extends the following result of R. F. Olin and J. E. Thomson: A compact subset K of the plane is the spectrum of an irreducible subnormal operator if and only if $\mathscr{R}(K)$ has exactly one nontrivial Gleason part G such that K is the closure of G. The main result of this paper is that the only additional requirement needed for the pair $\{K, K_e\}$ to be the spectrum and essential spectrum, respectively, is that K_e be a compact subset of K which contains the boundary of K. Additionally, results are obtained on the question of which index values can be specified on the various components of the complement of K_e.

Introduction and a survey of the results. All Hilbert spaces in this paper are separable and over the complex field. For a Hilbert space $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$ denotes the algebra of bounded linear operators on $\mathscr{H}$ and $\mathscr{B}_0(\mathscr{H})$ denotes the ideal of compact operators on $\mathscr{H}$. If $A \in \mathscr{B}(\mathscr{H})$, then $\sigma(A)$ is the *spectrum* of A and $\sigma_e(A)$ is the *essential spectrum* of A; that is $\sigma_e(A)$ is the spectrum of $A + \mathscr{B}_0(\mathscr{H})$ in $\mathscr{B}(\mathscr{H})/\mathscr{B}_0(\mathscr{H})$. If $\lambda \notin \sigma_e(A)$, then $A - \lambda$ is a *Fredholm* operator and $\operatorname{ind}(A-\lambda) = \dim\ker(A-\lambda) - \dim\ker(A-\lambda)^*$ is the *index*. See Douglas [5] for properties of the index. In particular, $\operatorname{ind}(A-\lambda)$ is constant on components of $\mathbf{C}\backslash\sigma_e(A)$.

An operator S on a Hilbert space $\mathscr{H}$ is *subnormal* if there is a normal operator N on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ such that $\mathscr{H}$ is invariant for N and S is the restriction of N to $\mathscr{H}$. Amongst all such normal extensions of S, there exists a *minimal normal extension* which is unique up to a unitary equivalence. This minimal normal extension of S will be denoted throughout this paper by N or $\operatorname{mne}(S)$. The operator S is pure if there is no reducing subspace for S on which S is normal. It is well known that if S is a pure subnormal operator, then $\operatorname{ind}(S-\lambda) \le -1$ for all $\lambda \in \sigma(S)\backslash\sigma_e(S)$.

If K is a compact subset of the plane, then $\mathscr{C}(K)$ denotes the space of continuous functions on K and $\mathscr{R}(K)$ the uniform closure in $\mathscr{C}(K)$ of the rational functions with poles off K. A Borel probability measure μ is a *representing measure* for $\mathscr{R}(K)$ at the point a in K if $f(a) = \int f\,d\mu$ for every f in $\mathscr{R}(K)$. Two points a and b in K belong to the same *Gleason part* of $\mathscr{R}(K)$ if there exists a constant $c > 0$ such that $1/c \le u(a)/u(b) \le c$ whenever u is the real part of a function f in $\mathscr{R}(K)$ and $u > 0$. A nontrivial Gleason part is a part which contains at least two distinct points. A fundamental result concerning $\mathscr{R}(K)$ is that if a and b belong to the same Gleason part and μ_a is a representing measure for $\mathscr{R}(K)$ at a, then there exists a representing measure μ_b for $\mathscr{R}(K)$ at b such that μ_a is absolutely

Received by the editors July 13, 1987 and, in revised form, December 29, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47B20; Secondary 47A10.

continuous with respect to μ_b and the support of μ_b is contained in the boundary of K together with the support of μ_a. For these and other facts concerning $\mathscr{R}(K)$ see Conway [3] or Gamelin [7].

If μ is a positive, regular, Borel measure supported on K, then $\mathscr{R}^2(K,\mu)$ is the closure of $\mathscr{R}(K)$ in the Hilbert space $L^2(\mu)$. If N_μ is the operator on $L^2(\mu)$ given by $(N_\mu f)(z) = zf(z)$, then $\mathscr{R}^2(K,\mu)$ is invariant for N_μ and $S_\mu = N_\mu|_{\mathscr{R}^2(K,\mu)}$ is a *rationally cyclic* subnormal operator.

If G is an open subset of $\mathbf{C}$, then $L^2_a(G)$ consists of those analytic functions $f\colon G \to \mathbf{C}$ such that

$$\iint_G |f(z)|^2 \, d\text{Area}(z) < \infty.$$

When endowed with the norm given by

$$\|f\|_{L^2_a(G)} = \left(\iint_G |f|^2 \, d\text{Area}\right)^{1/2},$$

$L^2_a(G)$ is a Hilbert space. For general properties of subnormal operators, rationally cyclic subnormal operators, and $L^2_a(G)$, the reader is referred to Conway [3].

If G is an open region in $\mathbf{C}$, then $H^2(G)$ consists of those analytic functions $f\colon G \to \mathbf{C}$ such that $|f|^2$ has a *harmonic majorant.* That is, there is a harmonic function $h\colon G \to [0,\infty)$ such that $|f(z)|^2 \le h(z)$ for all z in G. Amongst all such harmonic majorants, there is a least harmonic majorant h_f. If a is any fixed point in G, then $\|f\| = h_f(a)^{1/2}$. The space $H^2(G)$ is a Hilbert space when endowed with this norm. For general properties of $H^2(G)$ and its relationship to harmonic measure for G evaluated at a, the reader is referred to Fisher [6] or Conway [4].

Throughout this paper the notation $\overline{K}$, $\text{int}(K)$, and ∂K denotes the closure, interior, and boundary respectively of the set $K \subset \mathbf{C}$.

In Clancey and Putnam [2], it is shown that a necessary and sufficient condition for a compact set K to be the spectrum of a *pure* subnormal operator is that $\mathscr{R}(K \cap D) \neq \mathscr{C}(K \cap D)$ whenever D is an open disk intersecting K in a nonempty set. An analogous result for irreducible subnormal operators appears in Olin and Thomson [11]. Here it is shown that the compact set K is the spectrum of an *irreducible* subnormal operator if and only if $\mathscr{R}(K)$ has exactly one nontrivial Gleason part G and K is the closure of G. A difficulty with the Clancey-Putnam and Olin-Thomson results is that it is not possible to prescribe, or even describe, the essential spectrum of the subnormal operator constructed with the given spectrum K. This difficulty is partially removed in Hastings [9]. Here it is shown that the pair $\{K, K_e\}$ of compact sets consists of the spectrum and essential spectrum, respectively, of a *pure* subnormal operator if and only if K satisfies the Clancey-Putnam condition and $\partial K \subset K_e \subset K$.

In this paper, it is shown in Theorem 5 that the pair $\{K, K_e\}$ of compact sets consists of the spectrum and essential spectrum, respectively, of an *irreducible* subnormal operator if and only if K satisfies the Olin-Thomson condition and $\partial K \subset K_e \subset K$. Additionally it is shown that values of the index less than or equal to -2 can be prescribed on the various components of $K \backslash K_e$. Under what conditions the index value -1 can be prescribed is left as an open question.

The main results. The main tool that is used in the proof of Theorem 5 is Theorem 4.1 of McGuire [**10**]. For completeness this result is restated below as Lemma 1 and a brief sketch of the construction is indicated. Lemmas 2, 3, and 4 below are preliminary to Theorem 5.

LEMMA 1. *Let $1 \leq n \leq \infty$ and let $\{\mu_j\}_{j=0}^n$ be a collection of pairwise singular, positive, regular, Borel measures which are supported on a compact subset of* $\mathbf{C}$. *For each $j = 0, \ldots, n$ assume that:*

(a) *$\mathscr{H}_j$ is a subspace of $L^2(\mu_j)$ which is invariant for the operator N_j on $L^2(\mu_j)$ defined by $(N_j f)(z) = zf(z)$ and $\mathscr{H}_j$ contains a nonzero constant function;*

(b) *if S_j is the restriction of N_j to $\mathscr{H}_j$, then S_j is an irreducible subnormal operator with $N_j = \mathrm{mne}(S_j)$;*

(c) *there exists a bounded linear operator $A_j\colon \mathscr{H}_0 \to \mathscr{H}_j$ such that $A_j S_0 = S_j A_j$ and the range of A_j contains a nonzero constant function.*

Then there exists an irreducible subnormal operator S such that S is similar to $\bigoplus_{j=0}^n S_j$.

SKETCH OF THE CONSTRUCTION. Assume $n < \infty$. For each $j = 1, \ldots, n$, let Δ_j be a subset of the support of μ_j such that $\mu_j(\Delta_j) > 0$ and $\mu_j(\mathbf{C}\backslash\Delta_j) > 0$. Let $P_{\Delta_j}\colon L^2(\mu_j) \to L^2(\mu_j)$ be the projection given by $P_{\Delta_j} f = \chi_{\Delta_j} f$ where $f \in L^2(\mu_j)$ and χ_{Δ_j} denotes the characteristic function of Δ_j. The operator S is given by multiplication by z on the subspace $\mathscr{H}$ of $\bigoplus_{j=0}^n L^2(\mu_j)$ defined by

$$\mathscr{H} = \{(f, P_{\Delta_1} A_1 f + g_1, P_{\Delta_2} A_2 f + g_2, \ldots, P_{\Delta_n} A_n f + g_n)\colon \\ f \in \mathscr{H}_0 \text{ and } g_j \in \mathscr{H}_j,\ 1 \leq j \leq n\}.$$

LEMMA 2. *Let D be an open disc centered at λ of radius R, let A_n be the annulus $\{z\colon R(1 - 1/n) < |z - \lambda| < R(1 - 1/(n+1))\}$, let m be a fixed positive integer, and for each $j = 1, \ldots, m$, let $G_j = \bigcup_{k=0}^\infty A_{j+km}$. If f is an analytic function on D such that $f \in L^2_a(G_j)$, then $f \in L^2_a(D)$ and there is a constant C independent of f and R such that $\|f\|_{L^2_a(D)} \leq C\|f\|_{L^2_a(G_j)}$.*

PROOF. Since f is analytic on D, $M_r(f) = \int_0^{2\pi} |f(\lambda + re^{i\theta})|^2\, d\theta$ is an increasing function of r on $[0, R)$. For each $n = 1, 2, \ldots$, let $M_n(f)$ and $m_n(f)$ denote the supremum and infimum respectively of $\{M_r(f)\colon \lambda + r \text{ is in } A_n\}$. It is straightforward to verify that

$$\mathrm{Area}(A_n) = \frac{\pi R^2(2n^2 - 1)}{(n+1)^2 n^2} \leq \frac{18}{7}\frac{\pi R^2[2(n+1)^2 - 1]}{(n+2)^2(n+1)^2} = \frac{18}{7}\mathrm{Area}(A_{n+1})$$

and that the constant $\frac{18}{7}$ is independent of R. Hence

$$\iint_{A_n} |f|^2\, d\mathrm{Area} \leq M_n(f)\mathrm{Area}(A_n) \\ \leq \frac{18}{7} m_{n+1}(f)\mathrm{Area}(A_{n+1}) \leq \frac{18}{7}\iint_{A_{n+1}} |f|^2\, d\mathrm{Area}.$$

Thus, if $I_n = \iint_{A_n} |f|^2 \, d\mathrm{Area}$, then

$$\begin{aligned}
\iint_D |f|^2 \, d\mathrm{Area} &= \sum_{n=1}^{\infty} I_n \\
&= (I_1 + I_2 + \cdots + I_j) + \sum_{k=0}^{\infty} (I_{j+km+1} + I_{j+km+2} + \cdots + I_{j+(k+1)m}) \\
&\leq \left[\left(\frac{18}{7}\right)^{j-1} + \cdots + \left(\frac{18}{7}\right) + 1 \right] I_j \\
&\quad + \sum_{k=1}^{\infty} \left[\left(\frac{18}{7}\right)^{m-1} + \cdots + \left(\frac{18}{7}\right) + 1 \right] I_{j+km} \\
&\leq \sum_{k=0}^{\infty} \left(\left(\frac{18}{7}\right)^{m-1} + \left(\frac{18}{7}\right)^{m-2} + \cdots + 1 \right) I_{j+km} \\
&= C \iint_{G_j} |f|^2 \, d\mathrm{Area}
\end{aligned}$$

where C is the constant $1 + \frac{18}{7} + \cdots + (\frac{18}{7})^{m-1}$. $\square$

Let G be a bounded open set in $\mathbf{C}$, let $K_0 = \{z \in G\colon \mathrm{distance}(z, \partial G) \geq 1\}$, and, for $n \geq 1$, let $K_n = \{z \in G\colon 1/(n+1) \leq \mathrm{distance}(z, \partial G) < 1/n\}$. Note $G = \bigcup_{n=0}^{\infty} K_n$ and for each $n \geq 0$, $\overline{K}_n$ is a compact subset of G. It is well known (see Proposition 2.4 on page 54 of Fisher [**6**]) that for each $n \geq 0$ there is a constant $c_n > 0$ such that $|f(z)| \leq c_n \|f\|_{H^2(G)}$ for all f in $H^2(G)$ and z in $\overline{K}_n$. Hence if χ_{K_n} denotes the characteristic function of K_n and $w_1(z) = \sum_{n=0}^{\infty} (1/c_n^2) \chi_{K_n}(z)$, then for each f in $H^2(G)$,

$$\begin{aligned}
\iint_G |f(z)|^2 w_1(z) \, d\mathrm{Area}(z) &= \sum_{n=0}^{\infty} \iint_{K_n} |f(z)|^2 \frac{1}{c_n^2} \, d\mathrm{Area}(z) \\
&\leq \sum_{n=0}^{\infty} \iint_{K_n} \|f\|_{H^2(G)}^2 d\mathrm{Area}(z) \\
&= \|f\|_{H^2(G)}^2 \mathrm{Area}(G).
\end{aligned}$$

Now let $\{D_k\}_{k=1}^{\infty}$ be a sequence of pairwise disjoint, closed discs in G with $\mathrm{Area}(G) = \sum_{k=1}^{\infty} \mathrm{Area}(D_k)$. If λ_k and r_k are the center and radius respectively of the disc D_k, then for each $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$, let $A_{n,k} = \{z\colon r_k(1-1/n) < |z-\lambda_k| < r_k(1-1/(n+1))\}$. Fix a positive integer m and for each $j = 1, 2, \ldots, m$, let $G_j = \bigcup_{k=1}^{\infty} (\bigcup_{n=0}^{\infty} A_{j+nm,k})$. Note that for each disc D_k, G_j contains every mth annulus starting with the jth annulus. Let $w_2(z) = \exp(-1/\mathrm{distance}(z, \partial G))$ for each z in G and let μ_j be the measure given by $d\mu_j(z) = w_1(z) w_2(z) \chi_{G_j}(z) \, d\mathrm{Area}(z)$, where χ_{G_j} denotes the characteristic function of G_j. Let $\mathscr{R}(G) = \{r\colon r$ is a rational function with poles off $G\}$. Note that if $\lambda \in \partial G$, then $(z-\lambda)^n \in \mathscr{R}(G)$ for each integer n. Also note that the weight functions w_1 and w_2 are defined so that $(z-\lambda)^n \in L^2(\mu_j)$ for each integer n. Hence $\mathscr{R}(G) \subset L^2(\mu_j)$. Let $\mathscr{H}(G, \mu_j)$ be the closure of $\mathscr{R}(G)$ in $L^2(\mu_j)$ and let S_j denote the operator on $\mathscr{H}(G,\mu_j)$ given by $(S_jf)(z) = zf(z)$.

LEMMA 3. *If S_j is the operator on $\mathscr{H}(G,\mu_j)$ described above, then S_j is a subnormal operator such that $\sigma(S_j) = \mathrm{cl}(G)$, $\sigma_e(S_j) = \partial G$, and* $\mathrm{index}(S_j - \lambda) = -1$ *for λ in G. Moreover, if G is connected, then S_j is irreducible.*

PROOF. It is clear that S_j is a subnormal operator with minimal normal extension N_j on $L^2(\mu_j)$ given by $(N_jf)(z) = zf(z)$.

If $\lambda \in \partial G$, then λ is in the support of μ_j. Hence $\lambda \in \sigma(S_j)$. Also if $\lambda \in \partial G$, then since $(z-\lambda)^{-n} \in \mathscr{R}(G)$ for each integer n, $\mathscr{R}(G)$ is contained in the range of $S_j - \lambda$. Since $\mathscr{R}(G)$ is dense in $\mathscr{H}(G,\mu_j)$ and $\lambda \in \sigma(S_j)$, it follows that $\lambda \in \sigma_e(S_j)$. Thus $\partial G \subset \sigma_e(S_j)$.

Now assume $\lambda \in G$ and let D be a closed disc of radius R centered at λ and contained in G. Since D is closed and contained in G, $\delta = \mathrm{distance}(D, \partial G) > 0$. Let $\mathscr{F} = \{k\colon D_k \cap D \text{ is not empty}\}$. Since each disc D_k is closed and contained in G, $\delta_k = \mathrm{distance}(D_k, \partial G) > 0$ for all $k = 1, 2, \ldots$. Since at most finitely many of the discs $\{D_k\}_{k=1}^{\infty}$ have diameter greater than $\delta/2$, the distance from $\mathrm{cl}(\bigcup_{k\in\mathscr{F}} D_k)$ to ∂G is positive. Hence area measure and μ_j are boundedly mutually absolutely continuous when restricted to $G_j \cap (\bigcup_{k\in\mathscr{F}} D_k)$. Thus by Lemma 2, if $f \in \mathscr{H}(G,\mu_j)$, then $f \in L_a^2(\bigcup_{k\in\mathscr{F}} \mathrm{int}(D_k))$. Since $\mathrm{Area}(G) = \mathrm{Area}(\bigcup_{k=1}^{\infty} D_k)$, $\mathrm{Area}(D \setminus \bigcup_{k\in\mathscr{F}} D_k) = 0$. Thus $f \in L_a^2(D)$ and

$$|f(\lambda)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(\lambda + re^{i\theta})\, d\theta \right| \le \frac{1}{2\pi} \int_0^{2\pi} |f(\lambda + re^{i\theta})|\, d\theta.$$

Hence

$$\begin{aligned} |f(\lambda)| &\le \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} |f(\lambda + re^{i\theta})| r\, d\theta\, dr \\ &\le \frac{1}{\pi R^2} \left(\iint_D |f|^2\, d\mathrm{Area} \right)^{1/2} \left(\iint_D 1\, d\mathrm{Area} \right)^{1/2} = \frac{1}{R\sqrt{\pi}} \|f\|_{L_a^2(D)} \\ &\le \frac{1}{R\sqrt{\pi}} \|f\|_{L_a^2(\bigcup_{k\in\mathscr{F}} \mathrm{int}(D_k))} \le C\|f\|_{L_a^2(\bigcup_{k\in\mathscr{F}} (D_k \cap G_j))}, \end{aligned}$$

by Lemma 2. Thus there is a constant C such that $|f(\lambda)| \le C\|f\chi_D\|_{L^2(\mu_j)} \le C\|f\|_{\mathscr{H}(G,\mu_j)}$. Hence evaluation at λ is a bounded linear functional on $\mathscr{H}(G,\mu_j)$. Thus $L = \{f \in \mathscr{H}(G,\mu_j)\colon f(\lambda) = 0\}$ is a closed subspace with codimension one in $\mathscr{H}(G,\mu_j)$. Note that the range of $S_j - \lambda$ is contained in L. Also if $f \in L$, then $f(z) = (z-\lambda)g(z)$ where $g = f/(z-\lambda)$ is analytic on $G\setminus\{\lambda\}$ and λ is a removable singularity for g. Since $f \in \mathscr{H}(G,\mu_j)$, there is a sequence $\{r_n\}$ of rational functions in $\mathscr{R}(G)$ which converges to f in $\mathscr{H}(G,\mu_j)$. A standard limiting argument now shows that $\{(r_n(z) - r_n(\lambda))/(z-\lambda)\}$ converges to $(f(z) - f(\lambda))/(z-\lambda) = (f(z))/(z-\lambda) = g(z)$. Hence $g \in \mathscr{H}(G,\mu_j)$, $L = \mathrm{range}(S_j - \lambda)$, $S_j - \lambda$ is Fredholm, and $\mathrm{index}(S_j - \lambda) = -1$.

Let P denote the orthogonal projection of $L^2(\mu_j)$ onto $\mathscr{H}(G,\mu_j)$ and suppose R is a projection which commutes with S_j. It follows from a theorem of Bram (see Bram [**1**] or Proposition 11.6 on page 198 of Conway [**3**]) that there is a projection Q in the commutant of the von Neumann algebra generated by N_j such that Q commutes with P and R is the restriction of Q to $\mathscr{H}(G,\mu_j)$. Since N_j is cyclic, the commutant of the von Neumann algebra generated by N_j is equal to $\{M_\varphi : \varphi \in L^\infty(\mu_j)\}$ where M_φ denotes the multiplication operator $(M_\varphi f)(z) = \varphi(z)f(z)$. Hence, there is a characteristic function χ in $\mathscr{H}(G,\mu_j)$ such that $Rf = \chi f$ for all f in $\mathscr{H}(G,\mu_j)$. Since χ is analytic on G, it follows that if G is connected, then χ is either identically zero or identically one. Thus S_j is irreducible if G_j is connected. $\square$

The following "folk" lemma is well known but does not seem to appear in the literature. Since it is used repeatedly in the proof of Theorem 5, a brief sketch of its proof is included for the sake of completeness.

LEMMA 4. *If G is a bounded open region, $K = \overline{G}$, and ω is harmonic measure for G evaluated at the interior point a in G,. then $\mathscr{R}^2(K,\omega)$ is isometrically contained in $H^2(G)$.*

PROOF. It suffices to show that if f is analytic in a neighborhood of K, then $f \in H^2(G)$ and $\|f\|_{\mathscr{R}^2(K,\omega)} = \|f\|_{H^2(G)}$. If f is analytic in a neighborhood of K, then the Perron function u_f associated with $|f|^2$ is a harmonic majorant of $|f|^2$ in G and is given by $u_f(z) = \int |f(\varsigma)|^2\, d\omega_z(\varsigma)$ where ω_z denotes harmonic measure for G evaluated at z (see part a of Proposition 7.4 on p. 330 of Conway [4]). Moreover, if v is any other harmonic majorant of $|f|^2$, then $u_f(z) = \int |f(\varsigma)|^2\, d\omega_z(\varsigma) \leq \int v(\varsigma)\, d\omega_z(\varsigma) = v(z)$ since f is analytic in a neighborhood of K. Also, $\|f\|^2_{H^2(G)} = u_f(a) = \int |f(\varsigma)|^2\, d\omega(\varsigma) = \|f\|^2_{\mathscr{R}^2(K,\omega)}$. $\square$

THEOREM 5. *Let K be a compact subset of $\mathbf{C}$ such that $\mathscr{R}(K)$ has exactly one nontrivial Gleason part Ω and $K = \overline{\Omega}$. If K_e is a compact subset of K such that $\partial K \subset K_e$, then there exists an irreducible subnormal operator S with spectrum K and essential spectrum K_e. Moreover if $V_1, V_2, \ldots$ are the components of $K\backslash K_e$ and $\{a_n\}_{n=1}^\infty$ is a sequence of integers such that $a_n \leq -2$ for all n, then S may be chosen so that* $\operatorname{ind}(S-\lambda) = a_n$ *for $\lambda \in V_n$, $n = 1,2,\ldots$.*

PROOF. The first stage of the proof consists of the construction of an irreducible, rationally cyclic, subnormal operator S_μ such that $\sigma(S_\mu) = K$, $\sigma_e(S_\mu) = \partial K$, and $\operatorname{ind}(S_\mu - \lambda) = -1$ for $\lambda \in \operatorname{int}(K)$.

Let $\{\Omega_j\}_{j=1}^n$ denote the components of $\operatorname{int}(\Omega)$, the interior of Ω. (If $\operatorname{int}(\Omega)$ is empty, then this step may be omitted.) For each $j = 1,2,\ldots$, let w_j denote a point in Ω_j, and let ω_j denote harmonic measure for Ω_j evaluated at w_j. Let z_0 be a point in Ω. Since ω_j is a representing measure for $\mathscr{R}(K)$ at the point w_j and since z_0 and w_j belong to the same Gleason part of $\mathscr{R}(K)$, there exists a representing measure η_j for $\mathscr{R}(K)$ at z_0 such that $\omega_j \ll \eta_j$ and η_j is supported on ∂K (see Corollary 9.5 on page 348 of Conway [3] or Corollary 1.2 on page 143 of Gamelin [7]). If $n = \infty$, then let $\eta = \frac{1}{2}\sum_{j=1}^n(\eta_j/2^j)$. If $n < \infty$, then let $\eta = (1/2n)\sum_{j=1}^n \eta_j$. Since each η_j is a probability measure, $\eta(\mathbf{C}) = \frac{1}{2}$.

Let $\{z_j\}_{j=1}^m$ be a dense subset of $\Omega\backslash\operatorname{int}(\Omega)$. For each z_j, let ν_j be a representing measure for $\mathscr{R}(K)$ at z_0 such that ν_j is supported on $\{z_j\} \cup \partial K$ and $\delta_j \ll \nu_j$, where δ_j denotes the unit point mass measure at z_j. If $m < \infty$, then let $\nu = (1/2m)\sum_{j=1}^m \nu_j$. If $m = \infty$, then let $\nu = \frac{1}{2}\sum_{j=1}^m(\nu_j/2^j)$. Since each ν_j is a probability measure, $\nu(\mathbf{C}) = \frac{1}{2}$. Thus $\mu = \eta + \nu$ is a representing measure for $\mathscr{R}(K)$ at z_0. (If $\operatorname{int}(\Omega)$ is empty, then set $\mu = 2\nu$. If $\operatorname{int}(\Omega) = \Omega$, then set $\mu = 2\eta$.)

Note that if $f \in \mathscr{R}^2(K,\mu)$, then $f \in \mathscr{R}^2(\mathrm{cl}(\Omega_j),\omega_j)$ for each $j = 1,\ldots,n$. Hence $f \in H^2(\Omega_j)$ for each $j = 1,\ldots,n$ by Lemma 4. It is well known that if T is the operator given by multiplication by z on $H^2(\Omega_j)$, then $\Omega_j \subset \sigma(T)\backslash\sigma_e(T)$. Thus $\sigma_e(S_\mu) \subset \sigma_e(T) \subset \partial\Omega$ since $\mathscr{R}^2(K,\mu) \subset \mathscr{R}^2(\overline{\Omega}_j,\omega_j) \subset H^2(\Omega_j)$. Also $\partial K = \partial\Omega$ since $K = \overline{\Omega}$. Since $\partial K = \partial\Omega$ and the support of ν is $\Omega\backslash\mathrm{int}(\Omega)$, it follows that $\partial K \subset \sigma(S_\mu)$. Thus $\sigma(S_\mu) = K$ and $\sigma_e(S_\mu) = \partial K$. Since S_μ is rationally cyclic, $\mathrm{ind}(S_\mu - \lambda) = -1$ for all $\lambda \in \sigma(S_\mu)\backslash\sigma_e(S_\mu)$.

To show that S_μ is irreducible, let R be a projection which commutes with S_μ. Since $\mathrm{mne}(S_\mu)$ is cyclic, it follows from Bram's theorem (see Proposition 11.6 on page 198 of Conway [**3**]) that $R = M_\chi$ where χ is a characteristic function in $\mathscr{R}^2(K,\mu)$ and $M_\chi f = \chi f$ for each $f \in \mathscr{R}^2(K,\mu)$. If $f \in \mathscr{R}(K)$, then $f(z_0) = \int f\,d\mu = \langle f, 1\rangle$. Let $\{f_n\}$ be a sequence in $\mathscr{R}(K)$ which converges to χ in $L^2(\mu)$. Since $\langle f_n^2, 1\rangle = f_n(z_0)f_n(z_0) = \langle f_n, 1\rangle^2$ for each n, $\langle\chi,1\rangle = \langle\chi,1\rangle^2$. Thus $\langle\chi,1\rangle = 0$ or 1. Since $\langle\chi,1\rangle = \int\chi\,d\mu$ and μ is a probability measure, it follows that χ is either 0 or 1 identically. Thus $R = 0$ or 1 and S_μ is irreducible.

The next stage of the construction consists of producing a countable collection of subnormal operators which satisfies the hypothesis of Lemma 1 and which yields the prescribed spectral properties when Lemma 1 is applied. There are two steps in this stage. The first step involves arranging the essential spectrum and index values on the closure of $K\backslash K_e$. The second step involves placing the interior of K_e in the essential spectrum.

Let V_k be a component of $K\backslash K_e$. (If $K\backslash K_e$ is empty, then this step may be omitted.) Since $a_k \le -2$, $|a_k| - 1 > 0$. Let $S_{k,1}, S_{k,2},\ldots,S_{k,m}$ be the irreducible, effectually rationally cyclic, subnormal operators obtained by applying Lemma 3 with $G = V_k$ and $m = |a_k| - 1$. Hence $\sigma(S_{k,j}) = \overline{V}_k$, $\sigma_e(S_{k,j}) = \partial V_k$, and $\mathrm{ind}(S_{j,k} - \lambda) = -1$ for each λ in V_k and $j = 1,2,\ldots,|a_k|-1$. Let $\mu_{k,j}$ be the measure corresponding to the measure μ_j in Lemma 3. Thus $S_{k,j}$ is the operator given by multiplication by z on the space $\mathscr{H}(V_k,\mu_{k,j})$ and $\{\mu_{k,j}\}_{j=1}^m$ is a collection of pairwise singular measures satisfying for each $j = 1,\ldots,|a_k|-1$:

(1) $\int |f|^2\,d\mu_{k,j} \le \mathrm{Area}(V_k)\|f\|^2_{H^2(V_k)}$ for all $f \in H^2(V_k)$; and

(2) $\mu_{k,j}$ is singular with respect to μ.

Note that inequality (1) above follows from the way in which the weight functions w_1 and w_2 were defined just prior to the statement of Lemma 3. Also note that since V_k is a component of $K\backslash K_e$, $V_k \subset \Omega_i$ for some i. Thus by Lemma 4, $\mathscr{R}^2(K,\mu) \subset \mathscr{R}^2(\overline{\Omega}_i,\omega_i) \subset H^2(\Omega_i) \subset H^2(V_k) \subset \mathscr{H}(V_k,\mu_{k,j})$ and the operator $A_{k,j}\colon \mathscr{R}^2(K,\mu) \to \mathscr{H}(V_k,\mu_{k,j})$ defined by $A_{k,j}f = f$ is a bounded linear operator whose range contains the constants. Also $A_{k,j}S_\mu = S_{k,j}A_{k,j}$ for each k,j, and $T_1 = S_\mu \oplus (\bigoplus_{k,j} S_{k,j})$ satisfies $\sigma(T_1) = K$, $\sigma_e(T_1) = \partial(K_e)$, and for each k, $\mathrm{ind}(T_1 - \lambda) = a_k$ for $\lambda \in V_k$.

Let $\{B_k\}_{k=1}^\infty$ be a collection of pairwise disjoint open discs contained in $\mathrm{int}(K_e)$ such that $\overline{\mathrm{int}(K_e)} = \overline{\bigcup_{k=1}^\infty B_k}$. (If $\mathrm{int}(K_e)$ is empty, then omit this step.) If r_k is the radius of the disc B_k, then let $\{r_{k,j}\}_{j=1}^\infty$ be an enumeration of the rational numbers in the interval $(0, r_k)$ and let $B_{k,j}$ be the disc of radius $r_{k,j}$ which is concentric with B_k. Let $S'_{k,j}$ be the operator given by multiplication by z on $H^2(B_{k,j})$ and let $\mu'_{k,j}$ denote arclength measure on $\partial B_{k,j}$. Since $\mathrm{int}(K_e) \subset \Omega$, each B_k is contained in a component of Ω. By Lemma 4, the operator $A'_{k,j}\colon \mathscr{R}^2(K,\mu) \to H^2(B_{k,j})$ defined by $A'_{k,j}f = f$ is a bounded linear operator whose range contains the constants and $A'_{k,j}S_\mu = S'_{k,j}A'_{k,j}$. Also note that $\{\mu\} \cup \{\mu_{k,j}\}_{k,j} \cup \{\mu'_{k,j}\}_{k,j}$ is a collection of pairwise singular measures, and that $S'_{k,j}$ is an irreducible, cyclic, subnormal operator with $\sigma(S'_{k,j}) = \overline{B_{k,j}}, \sigma_e(S'_{k,j}) = \partial B_{k,j}$, and $\operatorname{ind}(S'_{k,j} - \lambda) = -1$ for $\lambda \in B_{k,j}$. Hence $T_2 = \bigoplus_{k,j} S'_{k,j}$ is a subnormal operator with spectrum and essential spectrum equal to $\overline{\operatorname{int}(K_e)}$.

The final stage in the construction is to apply Lemma 1 to the countable collection of operators $S_\mu \cup \{S_{k,j}\}_{k,j} \cup \{S'_{k,j}\}_{k,j}$ with $\mu_0 = \mu$, $S_0 = S_\mu$, and $\mathscr{H}_0 = \mathscr{R}^2(K, \mu)$. The irreducible subnormal operator S obtained is similar to $T_1 \oplus T_2$ and thus has the desired spectral properties. □

The first stage of the proof of Theorem 5 is of independent interest and is stated below as Corollary 6.

COROLLARY 6. *Let K be a compact subset of $\mathbf{C}$ such that $\mathscr{R}(K)$ has exactly one nontrivial Gleason part Ω and $K = \overline{\Omega}$. If z_0 is a point in G, then there is a representing measure μ for $\mathscr{R}(K)$ at the point z_0 such that the rationally cyclic subnormal operator S_μ on $\mathscr{R}^2(K, \mu)$ satisfies $\sigma(S_\mu) = K$ and $\sigma_e(S_\mu) = \partial K$.*

It is also worth remarking that if in Theorem 5 it is assumed that $\operatorname{Area}(K_e) = 0$ and $\sum_j a_j \operatorname{Area}(V_j) < \infty$, then the operator S has the additional property that $S^*S - SS^*$ is a trace class operator (see Corollary 9 of Hadwin and Nordgren [**8**]).

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