PHRAGMÉN-LINDELOF THEOREM FOR THE MINIMAL SURFACE EQUATION

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ABSTRACT. It is proved that if \( u \) satisfies the minimal surface equation in an unbounded domain \( \Omega \) which is properly contained in a half plane, then the growth property of \( u \) depends on \( \Omega \) and the boundary value of \( u \) only.

1. Introduction. The purpose of this paper is to establish a Phragmén-Lindelöf Theorem for the minimal surface equation in \( \mathbb{R}^2 \). We prove that if \( u \) satisfies the minimal surface equation in an unbounded domain \( \Omega \), which is properly contained in a half plane, then the growth property of \( u \) depends on \( \Omega \) and \( u|_{\partial \Omega} \) only, without requiring any other restriction for \( u \). In this respect, the Phragmén-Lindelöf Theorem for the minimal surface equation is better than that of the linear equations. We remark that if \( u \) satisfies the Laplace equation with vanishing boundary value in a sector domain \( \Omega_{\alpha} \) with angle \( 0 < \alpha < \pi \), then we cannot conclude that \( u \equiv 0 \). If we want to establish a maximum principle on \( \Omega_{\alpha} \), we must impose some restriction on the growth of \( u \) at infinity \([12, \text{Chapter 2, §9}]\). In fact, for the Laplace equation in an unbounded domain \( \Omega \), the growth property of \( u \) cannot be determined completely by \( u|_{\partial \Omega} \) alone. There are various types of restriction on the growth of \( u \) at infinity in Phragmén-Lindelöf Theorems for general equations. For these results, the reader is referred to \([1, 2, 3, 5, 6, 7, 10, 12, 13]\).

On the other hand, if \( u \) satisfies the minimal surface equation with vanishing boundary value in \( \Omega_{\alpha} \), then \( u \equiv 0 \) \([8, \text{p. 256}]\). So it is natural to conjecture that if \( u \) satisfies the minimal surface equation in \( \Omega_{\alpha} \), then the growth property of \( u \) depends on \( u|_{\partial \Omega} \) only.

We prove that if \( \Omega \subset \{(x, y) | y > 0, -\sqrt{\cosh y} < x < \sqrt{\cosh y} \} \) where \( f \in C^0[0, \infty) \), \( f \geq 0, f(t) \) increases as \( t \) increases, then the previous conjecture is true (Main Theorem). Our estimates depend on the shape of \( \Omega \), and the behavior of \( u|_{\partial \Omega} \) does not enter the picture explicitly.

We emphasize that in such a domain \( \Omega \), the solutions for the minimal surface equation with vanishing boundary value may not be unique, but the Main Theorem is still true. (e.g. in \( \Omega = \{(x, y) | -\sqrt{\cosh y}^2 - 1 < x < \sqrt{\cosh y}^2 - 1, y > 0 \} \), we have two solutions with vanishing boundary value \( u \equiv 0 \) and \( u = \sqrt{\cosh y}^2 - x^2 - 1 \)). Since in a half plane, the bound of the solutions with vanishing boundary value does not even exist, the domain must be properly contained in a half plane.

Some examples and remarks can be found in §3.
2. Main theorem. Throughout the whole article, $\Omega$ will be a connected domain (bounded or unbounded) in $\mathbb{R}^2$, and for any function $u \in C^1(\Omega)$, $Tu$ will denote the vector $Du/\sqrt{1 + |Du|^2}$ where $Du$ is the gradient vector of $u$.

For latter arguments of comparison, a particular function will be utilized. Its definition and basic properties are stated in the following lemma whose proof follows by direct computation:

**Lemma 2.1.** Let $v = (1/c) \cdot (x^2 - x_0^2)/(y - y_0) + a(y - y_0) + b$ in $\Omega = (-x_0, x_0) \times (-\infty, y_0)$ where $c, a, x_0$ are positive constants and $b, y_0$ are constants. Then we have

(i) $\lim_{y \to y_0} v = +\infty$,
(ii) $\operatorname{div}Tv \leq 0$ in $\Omega$ for $0 < c \leq 4a$,
(iii) $v(x, y) \geq v(x_0, y) = v(-x_0, y)$ for every $y < y_0$ and $x \in (-x_0, x_0)$.

Now, we have

**Theorem 2.2.** Let $\Omega \subset (-x_0, x_0) \times (0, y_0)$ and let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Suppose that

(i) $\operatorname{div}Tu \geq 0$ in $\Omega$,
(ii) $u|_{\partial\Omega \cap [-x_0, x_0] \times \{y\}} \leq ay + b$, where $a, x_0, y_0$ are positive constants, $b$ is a constant, $0 \leq y < y_0$.

Then if $y_0 - x_0/(2a) > 0$, we have $u(x, y) \leq ay_0 + b$ for every $(x, y) \in (-x_0, x_0) \times (0, y_0 - x_0/(2a)) \cap \Omega$.

**Proof.** Let $v = (1/(4a)) \cdot (x^2 - x_0^2)/(y - y_0) + ay + b$. Since $u|_{\partial\Omega} \leq v|_{\partial\Omega}$ and $\operatorname{div}Tu \geq 0 \geq \operatorname{div}Tv$ in $\Omega$, we have $u \leq v$ in $\Omega$. Noting that $v(x, y) \leq ay_0 + b$ for every $(x, y) \in (-x_0, x_0) \times (0, y_0 - x_0/(2a)) \cap \Omega$, one immediately completes the proof.

**Remark.** Since

$$v|_{(-x_0, x_0) \times \{y_0\}} = +\infty,$$

there are no restrictions for $u|_{\partial\Omega \cap [-x_0, x_0] \times \{y_0\}}$. The Main Theorem follows from this idea:

**Main Theorem.** Let $\Omega \subset \Omega_1 = \{(x, y) | y > 0, -f(y) < x < f(y)\}$ where $f, g \in C^0[0, \infty)$, $f, g \geq 0$, $g(0) = 0$, $f(t), g(t)/t$ increase as $t$ increases, and let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Suppose that

(i) $\operatorname{div}Tu \geq 0$ in $\Omega$,
(ii) $u|_{\partial\Omega \cap [-f(y), f(y)] \times \{y\}} \leq g(y)$ where $y \in [0, \infty)$,
(iii) $0 < \beta(y) \equiv f(y)/(2g(y)) < 1$ for some $y_1 > 0$ and every $y > y_1$,
(iv) $\beta(y)$ decreases in $[y_1, \infty)$.

Then $u(x, y) \leq g(y)/(1 - \beta(y))$ for every $(x, y) \in \Omega$ where $y > y_1$.

**Proof.** Fixing $y_2 > y_1$ and noting that for every $0 \leq y < y_2$, we have $u|_{\partial\Omega \cap [-f(y), f(y)] \times \{y\}} \leq g(y) \leq (g(y_2)/y_2)$. By Theorem 2.2, we have $u(x, y) \leq g(y_2)$ where $0 \leq y \leq y_2 - \beta(y_2)y_2$. Now, for every $y_3 > 1$, we have $y_3/(1 - \beta(y_3)) > y_3 > y_1$ and $y_3 = (1 - \beta(y_3))y_3/(1 - \beta(y_3)) \leq y_3/(1 - \beta(y_3)) - \beta(y_3)/(1 - \beta(y_3))y_3/(1 - \beta(y_3))$, and we obtain $u(x, y_3) < g(y_3/(1 - \beta(y_3)))$.

**Remark.** In the above theorem, $f$ controls the increasing rapidity of width of the defined domain $\Omega$ and $g$ controls the increasing rapidity of boundary value. It is natural that we require $g(y)/y$ to increase as $y$ increase.
3. Examples and remarks. Now, we give some examples for the applications of the Main Theorem.

**EXAMPLE 3.1.** Let \( \Omega = \{ -y < x < y \mid y > 0 \} \) and let \( u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \). Suppose that \( \text{div } Tu \geq 0 \) in \( \Omega \) and \( u(\pm y, y) \leq y^m \) for every \( y \geq 0 \), where \( m > 1 \) is a positive constant. Then for \( y \geq 1 \), we have \( \beta(y) = (2y^{m-1})^{-1} \leq 1 \), and \( u(x, y) \leq (y/(1 - (2y^{m-1})^{-1}m) = y^m + (m/2)y + O(y^{2-m}) \) as \( y \to \infty \).

**REMARK 3.2.** As [8, p. 256], with the help of the general maximum principle and a suitable solution of Dirichlet's problem [8, II, 7.2] the following maximum principle can be proved: Let \( \Omega = \{ -y < x < y \mid y > 0 \} \) (or a sector domain \( \Omega_\alpha \) with angle \( 0 < \alpha < \pi \)), and let \( u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \) be a solution of the minimal surface equation in \( \Omega \). Then

(i) if \( u(\pm y, y) < ay + c \), we have \( u(x, y) < ay + c \),
(ii) if \( u(\pm y, y) > ay + c \), we have \( u(x, y) > ay + c \),

where \( a \) and \( c \) are constants. Further,

(iii) if \( u(\pm y, y) \leq g(y) \) where \( g(y) \in C^0[0, \infty) \cap C^2(0, \infty) \) and \( g''(y) \leq 0 \) for every \( y > 0 \), we have \( u(x, y) \leq g(y) \),
(iv) if \( u(\pm y, y) \geq h(y) \) where \( h(y) \in C^0[0, \infty) \cap C^2(0, \infty) \) and \( h''(y) \geq 0 \) for every \( y > 0 \), we have \( u(x, y) \geq h(y) \).

In (iii), we have \( u(\pm y, y) \leq g(y_0) + g'(y_0)(y - y_0) \) for every fixing \( y_0 > 0 \), by (ii) \( u(x, y) \leq g(y_0) + g'(y_0)(y - y_0) \) and \( u(x, y_0) \leq g(y_0) \).

In Example 3.1, if \( u(\pm y, y) \leq y^m \) where \( 0 < m \leq 1 \) is a constant, by (iii) we have \( u(x, y) \leq y^m \), but the Main Theorem will not give the optimal result.

In Example 3.1, if \( u(\pm y, y) \leq y^m \) where \( 1 < m \) is a constant, by (iv) \( u(x, y) \) may be greater than \( y^m \), it seems that \( u(x, y) = y^m + (m/2)y + O(y^{2-m}) \) is a good estimate.

Now, we give two examples to explain how to estimate the solutions with vanishing boundary value.

**EXAMPLE 3.3.** Let \( \Omega = \{ -\sinh y < x < \sinh y \mid y > 0 \} \) and let \( u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \). Suppose that \( \text{div } Tu \geq 0 \) in \( \Omega \) and \( u|_{\partial \Omega} \leq 0 \). Consider \( g(y) = c \sinh y \), where \( c > \frac{1}{2} \) is a constant to be specified. Then \( \beta(y) = 1/2c < 1 \) and \( u(x, y) \leq c \sinh(y/(1 - (2c)^{-1})) \leq (\frac{c}{2})e^{y/(1-(2c)^{-1})} \). Let \( c = y/2 > 1 \), then

\[
\begin{align*}
   u(x, y) &\leq \frac{y}{4} \cdot e^{y/(1-y^{-1})} = \frac{y}{4} e^{y(1+\frac{1}{y}+\frac{1}{y^2}+\ldots)} \\
   &\leq (e + O(1/y)) \frac{y}{4} e^y \quad \text{as } y \to \infty.
\end{align*}
\]

**EXAMPLE 3.4.** In the case of catenoid, \( u = \sqrt{(\cosh y)^2 - x^2} \) and the defined domain \( \Omega = \{ -\cosh y < x < \cosh y \mid y > 0 \} \). Since \( \Omega \subset \{ -\sinh(y + 1) < x < \sinh(y + 1) \mid y > -1 \} \) and \( u|_{\partial \Omega} \leq c \sinh(y + 1) \), where \( c > 1 \). By Example 3.3, we have \( u(x, y) = O((y + 1) \cdot e^{y+1}) = O(y e^y) \).

**REMARK 3.5.** In Example 3.4, the actual growth behavior is \( \cosh y = O(e^y) \), and our estimate \( O(y e^y) \) are not optimal. In fact, the slover the growth of \( u \) on \( \partial \Omega \), the poorer the estimates of \( u \) in \( \Omega \).

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