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ON THE SPECTRAL RIGIDITY OF CP^n

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ABSTRACT. Complex projective space CP^n with the Fubini-Study metric has recently been characterized by the spectrum of the Laplacian on 2-forms. This important result was proved separately for $n \neq 2, 8$ by B. Y. Chen and L. Vanhecke, and for $n = 2$ and $n = 8$ by S. I. Goldberg. In this paper, we give a new proof which does not distinguish the three cases. It makes strong use of a result of S. Kobayashi and T. Ochiai, and may be applied to the spectrum of the Laplacian on 1-forms. Moreover, a characterization of CP^2 by the spectrum of the Laplacian on 1-forms is given.

1. Introduction. Let (M, g, J) be a Kaehler manifold of complex dimension n. It is assumed here and in the sequel that M is connected compact and of complex dimension $n > 1$. We denote by ${}^p\mathrm{Spec}(M, g)$ the spectrum of the real Laplacian, with respect to the Kaehler metric g, on p-forms on M. From Hodge theory we have that $0 \in {}^p\mathrm{Spec}(M, g)$ if and only if the pth Betti number $b_p(M) \neq 0$, and its multiplicity is then $b_p(M)$. Let (CP^n, g_0, J_0) be the n-dimensional complex projective space, where J_0 is the standard complex structure and g_0 is the Fubini-Study metric of constant holomorphic sectional curvature $c = 1$.

In [**2**] the following question was posed: "If ${}^p\mathrm{Spec}(M, g) = {}^p\mathrm{Spec}(CP^n, g_0)$ for some fixed p, $0 \leq p \leq 2n$, is (M, g, J) holomorphically isometric to (CP^n, g_0, J_0)?" The answer to this question is yes in the following cases.

(A) $p = 0$ and $n \leq 6$ or $p = 1$ and $8 \leq n \leq 51$ [**10, 11**];

(B) $p = 2$ for all n [**2, 3**];

(C) (M, g, J) is cohomologically Einstein, for all n and p with the following possible exceptions: (i) n and p satisfy the relation $p^2 - 2np + n(2n-1)/3 = 0$, and (ii) $p = 1$ or $2n - 1$ with $n \leq 7$ [**4**].

One purpose of this paper is to prove (B) without distinguishing the three cases $n \neq 2, 8$ [**2**], $n = 2$, and $n = 8$ [**3**]. The method we use to prove (B) (see §3) is different from the previous authors because the crucial point in our proof is a result of Kobayashi and Ochiai [**6**]. Our other purpose is to remove the case $p = 1$ (or 3) and $n = 2$ in (C). In fact, we prove the following:

THEOREM 1. *Let (M, g, J) be a cohomologically Einstein Kaehler manifold with ${}^p\mathrm{Spec}(M, g) = {}^p\mathrm{Spec}(CP^2, g_0)$ for $p = 1$ (or 3). Then (M, g, J) is holomorphically isometric with (CP^2, g_0, J_0).*

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2. Preliminaries. Let (M, g, J) be a Kaehler manifold of complex dimension n. Let $\theta^1 \cdots \theta^n$ be a local field of unitary coframes. Then the Kaehler metric g, the fundamental 2-form ϕ, the Ricci tensor ρ and the scalar curvature τ are given, respectively, by

$$g = \frac{1}{2}\sum(\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i), \qquad \phi = \frac{\sqrt{-1}}{8\pi}\sum \theta^i \wedge \bar{\theta}^i,$$
$$\rho = \frac{1}{2}\sum(\rho_{i\bar{j}}\theta^i \otimes \bar{\theta}^j + \bar{\rho}_{i\bar{j}}\bar{\theta}^i \otimes \theta^j), \qquad \tau = 2\sum \rho_{i\bar{i}},$$

where $\rho_{i\bar{j}} = 2\sum R^k_{ik\bar{j}}$ and $R^i_{jk\bar{h}}$ are the components of the curvature tensor R. We denote by $|R|$ and $|\rho|$ the lengths of R and ρ, respectively. Let

$$\gamma = (\sqrt{-1}/4\pi)\sum \rho_{i\bar{j}}\theta^i \wedge \bar{\theta}^j.$$

Then, the first Chern class c_1 of M is represented by γ. We say that M is *cohomologically Einstein* if $c_1 = a\omega$ for some real number a, where ω is the cohomology class represented by ϕ. It is known that (cf. [**8**]) if $c_1 = a\omega$ for some real number a, then

$$\text{(2.1)} \qquad \int_M \tau * 1 = an\,\mathrm{vol}(M, g) \text{ and } \int_M (\tau^2 - 2|\rho|^2) * 1 = n(n-1)a^2\,\mathrm{vol}(M, g).$$

3. A new proof of (B). Assume ${}^2\mathrm{Spec}(M, g) = {}^2\mathrm{Spec}(CP^n, g_0)$. Then

$$b_2(M) = b_2(CP^n) = 1.$$

Moreover by the asymptotic expansion of Minakshisundaram-Pleijel-Gaffney for ${}^2\mathrm{Spec}$, $\dim_C M = \dim_C CP^n = n$,

$$\text{(3.1)} \qquad \mathrm{vol}(M, g) = \mathrm{vol}(CP^n, g_0), \qquad \int_M \tau * 1 = \int_{CP^n} \tau_0 * 1,$$

and
(3.2)
$$\int_M \{5(2n^2 - 25n + 60)\tau^2 - 2(2n^2 - 181n + 540)|\rho|^2 + 2(n-8)(2n-15)|R|^2\} * 1$$
$$= \int_{CP^n} \{5(2n^2 - 25n + 60)\tau_0^2 - 2(2n^2 - 181n + 540)|\rho_0|^2$$
$$+ 2(n-8)(2n-15)|R_0|^2\} * 1$$

(cf. [**9**]), where τ_0, ρ_0, R_0 indicate the corresponding quantities for CP^n. Since $b_2(M) = 1$, the manifold M is a Hodge manifold (cf. [**7**, p. 37]), that is, there exists a positive element $\alpha \in H^{1,1}(M, Z)$. On the other hand, $b_2(M) = 1$ implies that $b_{1,1}(M)$, the dimension of the space of harmonic $(1,1)$-forms on M, is equal to 1. Therefore, multiplying the Kaehler metric g by some constant, if necessary, we may assume that $\omega = [\phi]$ is a positive element of $H^{1,1}(M, Z)$. Moreover, $b_2(M) = 1$ implies that M is cohomologically Einstein, i.e. $c_1 = a\omega$ for some real number a. Thus (2.1) and (3.1) imply

$$a = [1/n\,\mathrm{vol}(M, g)]\int_M \tau * 1 = [1/n\,\mathrm{vol}(CP^n, g_0)]\int_{CP^n} \tau_0 * 1 = n + 1.$$

Consequently, $c_1 = (n+1)\omega$, so by the Corollary to Theorem 1.1 in [**6**], it follows that M is biholomorphic to CP^n. Now let $\omega_0' = [\phi_0']$ be the Kaehler class corresponding to the Fubini-Study metric g_0' on M with constant holomorphic sectional curvature $c = 1$. Since $b_2(M) = 1$, we have $\omega = k\omega_0'$ for some $k \in R$. Since the volume depends only on the Kaehler class and $\mathrm{vol}(M, g_0') = \mathrm{vol}(CP^n, g_0) = \mathrm{vol}(M, g)$, we have $k = 1$ so that $\omega = \omega_0'$. Therefore, the second Chern class c_2 of M satisfies

$$\omega^{n-2}c_2 = \omega_0'^{n-2}c_2$$

and consequently (cf., for example, [**1**, p. 118]),

$$\int_M (\tau^2 - 4|\rho|^2 + |R|^2) * 1 = \int_M (\tau_0'^2 - 4|\rho_0'|^2 + |R_0'|^2) * 1 \tag{3.3}$$
$$= \int_{CP^n} (\tau_0^2 - 4|\rho_0|^2 + |R_0|^2) * 1,$$

where τ_0', ρ_0', R_0' denote the corresponding quantities on M defined by g_0'. Moreover, since M is cohomologically Einstein, (2.1) and (3.1) imply

$$\int_M (\tau^2 - 2|\rho|^2) * 1 = \int_{CP^n} (\tau_0^2 - 2|\rho_0|^2) * 1. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we obtain $\int_M |\rho|^2 * 1 = \int_{CP^n} |\rho_0|^2 * 1$ and $\int_M \tau^2 * 1 = \int_{CP^n} \tau_0^2 * 1$, so that

$$\int_M (|\rho|^2 - \tau^2/2n) * 1 = \int_{CP^n} (|\rho_0|^2 - \tau_0^2/2n) * 1. \tag{3.5}$$

Since (CP^n, g_0) is Einstein, we find from (3.5) that (M, g) is Einstein. Now, because M is Einstein and biholomorphic to CP^n, a result of Berger (cf. for example [**7**, p. 74]) implies

$$g = kg_0' \quad \text{for some constant } k > 0.$$

This, together with $\omega = \omega_0'$, implies that $g = g_0'$ and so (M, g, J) is holomorphically isometric to (CP^n, g_0, J_0).

REMARK. The second term of the asymptotic expansion for ${}^p\mathrm{Spec}(M, g)$ is given by (cf. [**9**])

$$a_{1,p} = \left\{ \frac{1}{6}\binom{2n}{p} - \binom{2n-2}{p-1} \right\} \int_M \tau * 1.$$

It follows that, if $p^2 - 2np + n(2n-1)/3 \neq 0$, then $a_{1,p}(M) = a_{1,p}(CP^n)$ implies $\int_M \tau * 1 = \int_{CP^n} \tau_0 * 1$. Consequently, by a similar argument as in the proof of (B), we may prove the following:

THEOREM 2. *If (M, g, J) is a Kaehler manifold with $b_2(M) = 1$, ${}^1\mathrm{Spec}(M, g) = {}^1\mathrm{Spec}(CP^n, g_0)$ and $n \neq 3$, then (M, g, J) is holomorphically isometric to (CP^n, g_0, J_0).*

Theorem 2 is an improvement of Corollary 4 of [**2**].

4. Proof of Theorem 1. Assume ${}^1\mathrm{Spec}(M, g) = {}^1\mathrm{Spec}(CP^2, g_0)$. Note that ${}^3\mathrm{Spec}(M, g) = {}^3\mathrm{Spec}(CP^2, g_0)$ implies ${}^1\mathrm{Spec}(M, g) = {}^1\mathrm{Spec}(CP^2, g_0)$. Then

$$b_1(M) = b_1(CP^2) = 0$$

and by the asymptotic expansion for ^{1}Spec, we obtain $\dim_C M = \dim_C CP^2 = 2$,

$$\mathrm{vol}(M,g) = \mathrm{vol}(CP^2, g_0), \qquad \int_M \tau * 1 = \int_{CP^2} \tau_0 * 1, \tag{4.1}$$

and

$$\int_M (11|R|^2 - 86|\rho|^2 + 20\tau^2) * 1 = \int_{CP^2} (11|R_0|^2 - 86|\rho_0|^2 + 20\tau_0^2) * 1 \tag{4.2}$$

(cf. [**9**]).

We recall that the signature of M is given by (cf. [**5**, p. 125])

$$\mathrm{sign}(M) = \sum_{p,q=0}^{2} (-1)^9 b_{p,q} = 2 + 4b_{2,0} - b_2,$$

where $b_{p,q}$ denotes the dimension of the space of harmonic forms of bidegree (p,q) on M, and $b_2 = b_{1,1} + b_{2,0} + b_{0,2}$. From (4.1), we have that the total scalar curvature of M is positive. Then a result of Yau [**12**] implies that the plurigenera P_m of M vanish. In particular, $b_{2,0} = P_1 = 0$. Therefore,

$$\mathrm{sign}(M) = 2 - b_2. \tag{4.3}$$

Moreover, the Euler characteristic of M is given by

$$\chi(M) = 2 - 2b_1 + b_2 = 2 + b_2. \tag{4.4}$$

From (4.3) and (4.4), we find

$$\mathrm{sign}(M) + \chi(M) = 4 = \mathrm{sign}(CP^2) + \chi(CP^2). \tag{4.5}$$

On the other hand, $\mathrm{sign}(M)$ and $\chi(M)$ are also given by

$$\mathrm{sign}(M) = (-1/48\pi^2) \int_M (|R|^2 - 2|\rho|^2) * 1,$$
$$\chi(M) = (1/32\pi^2) \int_M (|R|^2 + \tau^2 - 4|\rho|^2) * 1.$$

Therefore, from (4.5), we get

$$\int_M (|R|^2 - 8|\rho|^2 + 3\tau^2) * 1 = \int_{CP^2} (|R_0|^2 - 8|\rho_0|^2 + 3\tau_0^2) * 1. \tag{4.6}$$

Moreover, since M is cohomologically Einstein, as before we have

$$\int_M (\tau^2 - 2|\rho|^2) * 1 = \int_{CP^2} (\tau_0^2 - 2|\rho_0|^2) * 1. \tag{4.7}$$

So, from (4.2), (4.6), (4.7),

$$\int_M \left(|R|^2 - \frac{4}{3}|\rho|^2\right) * 1 = \int_{CP^2} \left(|R_0|^2 - \frac{4}{3}|\rho_0|^2\right) * 1 = 0. \tag{4.8}$$

Since $|R|^2 \geq (4/3)|\rho|^2$, from (4.8), we obtain $|R|^2 = (4/3)|\rho|^2$ and hence (M,g,J) is of constant holomorphic sectional curvature c. Finally (4.1) implies $c = 1$, so (M,g,J) is holomorphically isometric to (CP^2, g_0, J_0).

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