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ISOTHERMIC SURFACES AND THE GAUSS MAP

BENNETT PALMER

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ABSTRACT. We give a necessary and sufficient condition for the Gauss map of an immersed surface M in n-space to arise simultaneously as the Gauss map of an anti-conformal immersion of M into the same space. The condition requires that the lines of curvature of each normal section lie on the zero set of a harmonic function. The result is applied to a class of surfaces studied by S. S. Chern which admit an isometric deformation preserving the principal curvatures.

1. Introduction. The classical Gauss map of a surface in $\mathbf{R}^3$ assigns to a point the unit normal vector to the surface. For a surface in $\mathbf{R}^N$ the Gauss map assigns to a point the tangent plane which may be identified with a point in a quadric $Q^{N-2} \subset \mathbf{C}P^{N-1}$.

In recent years several papers have discussed the determination of a surface by its Gauss map. The results of K. Kenmotsu [**6**] show that a smooth map from a Riemann surface R to the 2-sphere, satisfying an integrability condition, factors through a conformal immersion

$$X: R \to M^2 \subset \mathbf{R}^3 \tag{1.1}$$

as the Gauss map. Kenmotsu's integrability condition explicitly involves the conformal structure of R and a real valued function h which is to be the mean curvature of M.

In [**3**] Hoffman and Osserman give conditions on a map

$$g: R \to Q^{N-2} \tag{1.2}$$

involving only the complex structure of R, which are necessary and sufficient for the map to arise as the Gauss map of a conformal immersion

$$X: R \to \mathbf{R}^N. \tag{1.3}$$

Their results essentially imply that a conformal immersion of a Riemann surface R into $\mathbf{R}^N$ is determined by its Gauss map, provided the mean curvature is not identically zero at some point.

Since a single map as in (1.2) determining multiple conformal immersions is in general excluded, a natural question to consider is when such a map arises simultaneously as the Gauss map of both a conformal and anti-conformal immersion. This question turns out to have a simple geometric answer which requires the following definition:

DEFINITION. A surface $M \subset \mathbf{R}^N$ is *isothermic* if, locally, there exist a pair of harmonic functions u_1, u_2 such that the lines of curvature of each smooth section of

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the normal bundle are contained in a level set $u_j = \text{const}$. This is a generalization of a classical definition which can be found in [**2**].

THEOREM I. *Let*

$$g: R \to Q^{N-2} \tag{1.4}$$

be the Gauss map of a conformal immersion of an orientable surface M :

$$X: R \to M \subset \mathbf{R}^N. \tag{1.5}$$

Then there exists an anti-conformal immersion

$$\tilde{X}: R \to \tilde{M} \subset \mathbf{R}^N \tag{1.6}$$

such that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{g} & Q^{N-2} \\ & \tilde{X}\searrow \quad \nearrow \text{Gauss Map} & \\ & \tilde{M} & \end{array}$$

if and only if M *is isothermic. The surface* $\tilde{M}$ *is unique (modulo similarity transformations of* $\mathbf{R}^N$*) provided* M *is not totally umbilic.*

There are abundant examples of isothermic surfaces. We list a few.
(1) Constant mean curvature surfaces in $\mathbf{R}^3$.
(2) Surfaces of revolution in $\mathbf{R}^3$.
(3) Constant mean curvature surfaces $M^2 \subset S^3 \subset \mathbf{R}^4$.
In addition, we note the following properties of isothermic surfaces which when combined with the above furnish more examples:
(1) $f(M)$ is isothermic if M is isothermic and $f: \mathbf{R}^N \to \mathbf{R}^N$ is conformal.
(2) If $X: R \to M \subset \mathbf{R}^N$ is isothermic then

$$\underbrace{X \oplus X \oplus \cdots \oplus X}_{k}: M \to \mathbf{R}^{Nk}$$

is isothermic.

In part 4 we apply the main result to a class of surfaces studied by S. S. Chern in [**1**]. These are surfaces of nonzero mean curvature which admit a nontrivial isometric deformation preserving the principal curvatures. We show first that these surfaces are isothermic and second that the surface $\tilde{M}$ described above has the property that is mean curvature is the reciprocal of a harmonic function.

2. Preliminaries. Let R be a simply connected Riemann surface and

$$X: R \to M \subset \mathbf{R}^3 \tag{2.1}$$

a smooth, conformal immersion. We assume M is orientable. After choosing a complex coordinate $z = u_1 + iu_2$ on R, the metric induced by (2.1) has an expression

$$ds_M^2 = e^{\rho}|dz|^2 \tag{2.2}$$

for a smooth function $\rho = \rho(z)$ on R. Locally on M we may choose an orthonormal frame $\{\xi^r\}_{r=3,\ldots,N}$ for the normal bundle $N(M)$. Differentiating ξ^r defines

$$d\xi^r = -A_r(\cdot) + \nabla^{\perp}_{(\cdot)}\xi^r. \tag{2.3}$$

The terms on the right-hand side are respectively the tangential and normal components of $d\xi^r$. At each point of M, $-A_r$ is a selfadjoint endomorphism of the tangent plane. Its eigenvalues β_j^r; $j = 1, 2$ are the principal curvatures of ξ^r. The corresponding eigenvectors are the principal directions and their integral curves are the lines of curvature. These curves exist away from the ξ^r umbilics which are those points of M where $\beta_1^r = \beta_2^r$.

The second fundamental forms are defined by

$$\Pi_p^r(X, Y) = ds_M^2(X, A_r(Y)), \qquad X, Y \in T_pM. \tag{2.4}$$

The trace of Π^r is twice the rth mean curvature

$$h^r = \tfrac{1}{2}(\beta_1^r + \beta_2^r). \tag{2.5}$$

On R, Π^r has an expression

$$\Pi^r = 2\,\mathrm{Re}\left(\frac{\phi^r}{2} dz \otimes dz + h^r \frac{e^\rho}{2} dz \otimes d\bar{z}\right). \tag{2.6}$$

The quantities

$$q^r \equiv \phi^r dz \otimes dz \tag{2.7}$$

define invariant quadratic differentials on M. Under change of complex coordinate

$$z_1 = z_1(z), \qquad \frac{dz_1}{d\bar{z}} = 0, \tag{2.9}$$

q^r transforms according to

$$q^r = \phi^r dz \otimes dz = \phi^r \left(\frac{dz}{dz_1}\right)^2 dz_1 \otimes dz_1; \tag{2.10}$$

that is

$$\phi_1^r = \phi^r \left(\frac{dz}{dz_1}\right)^2. \tag{2.11}$$

The zeros of q^r are precisely the ξ^r umbilics. See [**5**] for details.

The quantities defined above appear as coefficients of the structural equations for the immersion

$$\begin{cases} X_{zz} = \rho_z X_z + \frac{1}{2}\sum_r \phi^r \xi^r, \\ X_{z\bar{z}} = \frac{e^\rho}{2}\sum_r h^r \xi^r, \\ \xi_z^r = -h^r X_z - \phi^r e^{-\rho} X_{\bar{z}} + \sum_t S_r^t \xi^t, \end{cases} \tag{2.12}$$

(and their conjugates) where S_r^t are defined by

$$\nabla_z^\perp \xi^r = \sum_{t=3}^N S_r^t \xi^t. \tag{2.13}$$

The well-known integrability conditions of (2.12) are the Gauss equation

$$\rho_{z\bar{z}} = \frac{1}{2}\sum_r (\phi^r \overline{\phi^r} e^{-\rho} - (h^r)^2 e^\rho), \tag{2.14}$$

the Codazzi equations

$$(\phi^r)_{\bar{z}} + \sum_t \phi^t(\overline{S_t^r}) = e^\rho (h^r)_z + \sum_t e^\rho h^t S_t^r \tag{2.16}$$

and the Ricci equations

$$\operatorname{Im}\left\{ (S_r^t)_{\bar{z}} - \frac{e^{-\rho}}{2}\phi^r \overline{\phi^t} + \sum_{l=3}^{N} S_r^l \overline{S_l^t} \right\} = 0, \qquad 3 \le r, t \le N. \tag{2.17}$$

We define the Gauss map of

$$X: R \to M \subset \mathbf{R}^N \tag{2.18}$$

following [**2**]. Consider the quadratic

$$Q^{N-2} = \{[\varsigma] \in \mathbf{C}P^{N-1} | \varsigma \cdot \varsigma = 0\}.$$

Here [ ] denotes equivalence class. Since X is conformal, one has

$$X_z \cdot X_z = 0 \tag{2.19}$$

and we define the Gauss map

$$\begin{aligned} g&: M \to Q^{N-2}, \\ p &\to [X_z], \end{aligned} \tag{2.20}$$

where p is a point on M with coordinate z. We will also use g to denote this map pulled back to R via X. Many details and interesting properties of g may be found in [**3** and **4**].

3. Main result. The proposition below gives simple coordinate-dependent criteria for a surface to be isothermic.

PROPOSITION 3.1. *M is isothermic if and only if locally there exists an isothermal parameter $z = u + iu_2$ with*

$$\operatorname{Im} \phi^r(z) = 0, \quad r = 3, \ldots, N. \tag{3.1}$$

PROOF. Assume (3.1) holds and write

$$\Pi^r = \sum_{i,j=1,2} L_{ij}^r du_i \otimes du_j. \tag{3.2}$$

An easy computation shows

$$\phi^r = \frac{L_{11} - L_{22}}{2} - iL_{12} \tag{3.3}$$

so that (3.2) implies

$$\Pi^r = L_{11}^r du_1 \otimes du_1 + L_{22}^r du_2 \otimes du_2 \tag{3.4}$$

and the lines of curvature are the coordinate curves.

On the other hand suppose u_1, u_2 are harmonic functions such that the lines of curvature are contained in a level set $\{u_j = \text{const.}\}$. Since the lines of curvature intersect orthogonally, u_1 and u_2 are harmonic conjugates. Define

$$z_1 = u_1 \pm iu_2, \tag{3.5}$$

the sign chosen so that $z_1 = z_1(z)$ is holomorphic. The lines of curvature are the solutions of

$$\operatorname{Im} \phi^r dz^2 = 0 \tag{3.6}$$

(see [5] for details). Since ∇u_1 (resp. ∇u_2) is tangent to the level curves $u_2 =$ const. (resp. $u_1 =$ const.), (3.6) implies

$$\operatorname{Im} \phi^r dz \otimes dz(\nabla u_j, \nabla u_j) = 0. \tag{3.7}$$

Using

$$\nabla u_j = 2e^{-\rho}(u_{j\bar{z}} X_z + u_{jz} X_{\bar{z}}), \qquad j = 1, 2, \tag{3.8}$$

this gives

$$\operatorname{Im} \phi(u_{j\bar{z}})^2 = 0, \qquad j = 1, 2. \tag{3.9}$$

The Cauchy-Riemann equations applied to u_1, u_2 give

$$u_{1\bar{z}} = -iu_{2\bar{z}} \tag{3.10}$$

and we find

$$\begin{aligned} 0 &= \operatorname{Im} \phi^r \cdot (u_{1\bar{z}}^2 - u_{2\bar{z}}^2 \pm 2u_{1\bar{z}}^2) \\ &= \operatorname{Im} \phi^r \cdot (u_{1\bar{z}}^2 - u_{2\bar{z}}^2 \mp 2iu_{1\bar{z}} u_{2\bar{z}}) \\ &= \operatorname{Im} \phi^r \cdot (u_{1\bar{z}} \mp iu_{2\bar{z}})^2 \\ &= \operatorname{Im} \phi^r \left(\frac{d\bar{z}_1}{d\bar{z}} \right)^2 \\ &= \operatorname{Im} \phi^r \left(\frac{dz}{dz_1} \right)^2 \left| \frac{dz_1}{dz} \right|^2 . \end{aligned}$$

Therefore

$$\operatorname{Im} \phi^r \left(\frac{dz}{dz_1} \right)^2 = 0 \tag{3.11}$$

and by Proposition (3.1) the coordinate $u_1 \pm iu_2$ is as required.

PROOF OF THEOREM I. Let

$$g: R \to Q^{N-2} \subset \mathbb{C}P^{N-1} \tag{3.12}$$

represent the Gauss map of

$$X: R \to M \subset \mathbf{R}^N. \tag{3.13}$$

The existence of $\tilde{X}$ is equivalent to the existence of a smooth $\mathbb{C}$-valued function f on R such that

$$e^f X_z = \tilde{X}_{\bar{z}} \tag{3.14}$$

i.e.,

$$[\tilde{X}_{\bar{z}}] = [X_z] \in Q^{N-2}. \tag{3.15}$$

The differential equation (3.14) for $\tilde{X}$ will be integrable provided

$$\operatorname{Im}(\partial_z e^f X_z) = 0. \tag{3.16}$$

Using (2.12) this becomes

$$0 = \operatorname{Im}\left\{e^f(f_z+\rho_z)X_z + e^f\sum_r \frac{1}{2}\phi^r\xi^r\right\}$$
$$= \operatorname{Im}\left\{(f_z+\rho_z)\tilde{X}_{\bar{z}} + e^f\sum_r \frac{1}{2}\phi^r\xi^r\right\}.$$

Note that the structural equations (2.12) applied to the immersion $\tilde{X}$ imply

$$f_z + \rho_z = 0 \tag{3.17}$$

so we can write

$$f = \bar{g} - \rho \tag{3.18}$$

with $g_{\bar{z}} = 0$. (g is holomorphic.) This gives for all r,

$$\operatorname{Im} e^{\bar{g}}\phi^r = 0. \tag{3.19}$$

Define a new isothermal coordinate

$$z_1 = \int^z e^{g/2}d\xi. \tag{3.20}$$

Then the transformation rule (2.11) for ϕ^r gives

$$\phi_1^r = e^{-g}\phi^r \tag{3.21}$$

and so (3.19) implies

$$\operatorname{Im}\phi_1^r = 0 \tag{3.22}$$

and M is isothermic.

Conversely if $\operatorname{Im}\phi^r = 0$ for all r then define

$$\tilde{X}_{\bar{z}} = e^{-\rho}X_z \tag{3.23}$$

and one easily checks using (2.12) that

$$\operatorname{Im}\partial_z(e^{-\rho}X_z) = \operatorname{Im}\left(e^{-\rho(1/2)}\sum_r \phi^r\xi^r\right) = 0. \tag{3.24}$$

For the uniqueness of $\tilde{M}$, note that by [**4**] Theorem 2.3 the Gauss map of $\tilde{M}$,

$$\tilde{g} = [\tilde{X}_{\bar{z}}] = [e^f X_z] \tag{3.25}$$

determines the immersion $\tilde{X}$ (mod similarities) provided $\tilde{X}_{\bar{z}z} \not\equiv 0$. However,

$$\tilde{X}_{\bar{z}z} = e^f\left((f_z+\phi_z)X_z + \sum_r \frac{\phi^r}{2}\xi^r\right).$$

The right-hand side cannot vanish identically unless $\phi^r \equiv 0$, $r = 3,\ldots,N$, which is the totally umbilic case.

4. Application to a class of surfaces. In [**1**] S. S. Chern classified the umbilic free surfaces $M \subset \mathbf{R}^3$ which admit a nontrivial isometric deformation preserving the principal curvatures. Besides the classical examples of constant mean curvature surfaces, Chern found a second class of Weingarten surfaces with the property that the conformal metric

$$d\hat{s}^2 = \|\nabla h\|^2 (h^2 - K)^{-1} ds_M^2$$

has constant Gaussian curvature $\hat{K} = -1$. We will show these surfaces are isothermic and show the surfaces $\tilde{M}$ obtained by Theorem I have an interesting property.

THEOREM II. *Let M be an umbilic free, nonminimal, surface in $\mathbf{R}^3$ admitting a nontrivial isometric deformation preserving the principal curvatures. Then*

(i) *M is isothermic*;

(ii) *If $\tilde{M}$ is the surface obtained via Theorem* I, *the mean curvature $\tilde{h}$ of $\tilde{M}$ satisfies*

$$\tilde{\Delta}\left(\frac{1}{\tilde{h}}\right) = 0 \quad (\tilde{\Delta} \textit{ is the Laplace-Beltrami operator on } \tilde{M}). \tag{4.1}$$

REMARK. If $h \equiv$ const. on M then the above result is well known. The surface $\tilde{M}$ also has constant mean curvature so (4.1) holds trivially.

Following [**1**], introduce an orthonormal moving frame $\{e_1, e_2, e_3\}$ along M with e_1, e_2 the principal directions and e_3 the unit normal. Let ω_j be the dual one forms and as usual define ω_{ij} by

$$d\omega_i = \omega_{ij} \wedge \omega_j. \tag{4.2}$$

By choice of frame we have:

$$\omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2 \tag{4.3}$$

where $a > c$ are the principal curvatures. Next introduce the one form

$$\theta_1 = \frac{2dh}{a-c} \tag{4.4}$$

Let α_1 be the symmetry of θ_1 with respect to the principal directions:

$$\alpha_1 = \theta_1 - 2\left(\frac{2}{a-c} e_2(h)\right)\omega_2. \tag{4.5}$$

Let $*$ denote the Hodge star operator

$${}^*\omega_1 = \omega_2, \qquad {}^*\omega_2 = -\omega_1 \tag{4.6}$$

and define

$$\theta_2 = {}^*\theta_1, \qquad \alpha_2 = {}^*\alpha_1. \tag{4.7}$$

The Codazzi equations on M can be used to show (see [**1**])

$$d\log(a-c) = \alpha_1 + 2{}^*\omega_{12}. \tag{4.9}$$

In addition we note the important formulas

$$\begin{aligned} &\text{(a)} \quad d\alpha_1 = 0, \\ &\text{(b)} \quad d\alpha_2 \wedge \alpha_2, \end{aligned} \tag{4.10}$$

which gives $\hat{K} = -1$ in the case $h \not\equiv$ const.

PROOF OF THEOREM III. We first show M is isothermic. By (4.9) and (4.10a) we have

$$d^*\omega_{12} = 0. \tag{4.12}$$

Locally we can define a function ρ by

$$^*\omega_{12} = \frac{-d\rho}{2}. \tag{4.13}$$

Define

$$du_j = e^{-\rho/2}\omega_j, \qquad j = 1, 2. \tag{4.14}$$

Then (4.13) together with the structural equations (4.2) imply $ddu_j = 0$. The coordinates u_1, u_2 have the necessary property of Proposition 3.1 and M is isothermic.

We assume by the remark that $h \neq \text{const}$. Using (3.23) we compute fundamental quantities on $\tilde{M}$ denoted with "tilde",

$$\begin{aligned} &\text{(i) } \tilde{\omega}_1 = e^{-\rho}\omega_1, \qquad \tilde{\omega}_2 = -e^{-\rho}\omega_2, \\ &\text{(ii) } \tilde{a} = e^{\rho}a, \qquad \tilde{c} = -e^{\rho}c, \\ &\text{(iii) } \tilde{h} = e^{\rho}\left(\frac{a-c}{2}\right). \end{aligned} \tag{4.15}$$

The Hodge star operator of $\tilde{M}$ gives

$$\tilde{*}\tilde{\omega}_1 = \tilde{\omega}_2, \qquad \tilde{*}\tilde{\omega}_2 = -\tilde{\omega}_1 \tag{4.16}$$

which gives using (4.15(i))

$$\tilde{*}\omega_1 = -\omega_2, \qquad \tilde{*}\omega_2 = \omega_1. \tag{4.17}$$

By (4.15(iii)), equation (4.1) is equivalent to

$$0 = d\tilde{*}d(e^{-\rho}(a-c)^{-1}).$$

Compute

$$\begin{aligned} d\tilde{*}d(e^{-\rho}(a-c)^{-1}) &= -d\tilde{*}e^{-\rho}[(a-c)^{-1}d\rho - (a-c)^{-1}d\log(a-c)] \\ &= -d\tilde{*}e^{-\rho}[(a-c)^{-1}d\rho - (a-c)^{-1}(\alpha_1 - d\rho)] \quad \text{(by (4.9), (4.13))} \\ &= d\tilde{*}e^{-\rho}(a-c)^{-1}\alpha_1 \\ &= -de^{-\rho}(a-c)^{-1}\alpha_2 \quad \text{(by (4.5), (4.18))} \\ &= e^{-\rho}[(a-c)^{-1}d\rho \wedge \alpha_2 + (a-c)^{-1}d \\ &\qquad \cdot \log(a-c) \wedge \alpha_2 + (a-c)^{-1}d\alpha_2] \\ &\qquad\qquad \text{(by (4.9), (4.10b))} \\ &= e^{-\rho}[(a-c)^{-1}d\rho \wedge \alpha_2 + (a-c)^{-1}d \\ &\qquad \cdot (\alpha_1 - d\rho) \wedge \alpha_2 - (a-c)^{-1}\alpha_1 \wedge \alpha_2] \\ &= 0. \end{aligned}$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CALIFORNIA 95064