THE FIRST DIRICHLET EIGENVALUE AND RADIUS OF A GEODESIC BALL

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ABSTRACT. We give certain relation between the first Dirichlet eigenvalue and radius of a geodesic ball in a connected, compact n-dimensional Riemannian globally symmetric space of rank one.

Let \( M \) be a connected, compact n-dimensional Riemannian globally symmetric space of rank one, \( L \) the diameter of \( M \), \( A(r) \) the surface area of a sphere with radius \( r \) (0 < \( r < L \)), in \( M \), \( \Delta \) the Laplace-Beltrami operator on \( M \), and \( B(r) \) denote a geodesic ball with radius \( r \), in \( M \).

Consider the following Dirichlet eigenvalue problem:

\[ \Delta f + \lambda f = 0 \text{ in } B(r), \quad f = 0 \text{ on } \partial B(r). \]

Let \( \lambda = \lambda(M, n, r) \) be the first Dirichlet eigenvalue of \( B(r) \). By domain monotonicity of Dirichlet eigenvalues, for fixed \( n \), \( \lambda(M, n, r) : (0, L) \rightarrow (0, \infty) \) is strictly decreasing. For our convenience, we introduce \( r = r(M, n, \lambda) \) which is the inverse function of \( \lambda(M, n, r) \).

**THEOREM.** If \( \lambda(M, n, r) = 2k\alpha(n + 2\beta + 2k) \) (\( k = 1, 2, 3, \ldots \)), then \( r = r(M, n, \lambda) \) is the first positive zero of

\[
T(t) = 1 + \sum_{j=1}^{k} (-\sin^2 \sqrt{\alpha t})^j \prod_{m=0}^{j-1} \frac{(k - m)(n + 2\beta + 2k + 2m)}{(m + 1)(n + 2m)},
\]

where \( \alpha \) and \( \beta \) are constants determined by \( M \). (See Table 1.) In particular,

\[
\sqrt{ar}(M, n, 2\alpha(n + 2\beta + 2)) = \arcsin \sqrt{n/(n + 2\beta + 2)},
\]

\[
\sqrt{ar}(M, n, 4\alpha(n + 2\beta + 4)) = \arcsin \sqrt{(n + 2 - \sqrt{(4\beta + 8)(n + 2)/(n + 2\beta + 4))}/(n + 2\beta + 6)}.
\]

**PROOF OF THE THEOREM.** It is well known that \( M \) is isometric to one of the following spaces: \( S^m(\alpha) \) (the sphere with constant curvature \( \alpha \)), \( P^m(\alpha) \) (the real projective space with constant curvature \( \alpha \)), \( CP^m(\alpha) \) (the complex projective space with constant holomorphic sectional curvature \( 4\alpha \)), \( QP^m(\alpha) \) (the quaternionic space with maximum sectional curvature \( 4\alpha \)), and \( Cay P^2(\alpha) \) (the Cayley plane with maximum sectional curvature \( 4\alpha \)), and also well known that

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$A(r) = C(\sin \sqrt{\alpha}r)^{n-1}(\cos \sqrt{\alpha}r)^{2\beta+1}$ for some constant $C$ determined by $M$. (See [3].)

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L$</th>
<th>$\beta$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^m(\alpha)$</td>
<td>$\pi/\sqrt{\alpha}$</td>
<td>$-1/2$</td>
<td>$m$</td>
</tr>
<tr>
<td>$P^m(\alpha)$</td>
<td>$\pi/2\sqrt{\alpha}$</td>
<td>$-1/2$</td>
<td>$m$</td>
</tr>
<tr>
<td>$C P^m(\alpha)$</td>
<td>$\pi/2\sqrt{\alpha}$</td>
<td>$0$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$Q P^m(\alpha)$</td>
<td>$\pi/2\sqrt{\alpha}$</td>
<td>$1$</td>
<td>$4m$</td>
</tr>
<tr>
<td>Cay $P^2(\alpha)$</td>
<td>$\pi/2\sqrt{\alpha}$</td>
<td>$3$</td>
<td>$16$</td>
</tr>
</tbody>
</table>

| TABLE 1 |

Note that we may take $\alpha = 1$. Since $M$ is two-point homogeneous, the eigenfunction of $\lambda(M, n, r)$ is radial. In geodesic polar coordinates on $M$, the radial part of $\Delta$ is

$$\Delta_r = \frac{d^2}{dt^2} + \left(\frac{dA}{dt}/A(t)\right) \frac{dt}{dt} \quad (0 < t < L).$$

Now the eigenfunction $T(t)$ of $\lambda(M, n, r)$ satisfies

$$T'' + [(n - 1) \cot t - (2\beta + 1) \tan t]T'(t) + \lambda T(t) = 0.$$ 

Solving ODE we have

$$T(t) = F(a, b; n/2; \sin^2 t) \quad (t \leq r < \pi/2),$$

where $F$ is the Gauss hypergeometric function, $a = (n + 2\beta + \sqrt{(n + 2\beta)^2 + 4\lambda})/4$, and $b = (n + 2\beta - \sqrt{(n + 2\beta)^2 + 4\lambda})/4$. Finally, we obtain

$$T(t) = 1 + \sum_{j=1}^{\infty} (\sin^2 t/2)^j \prod_{m=1}^{j} \frac{-\lambda + 2(m - 1)(n + 2\beta + 2m - 2)}{m(n + 2m - 2)} \quad (t \leq r < \pi/2).$$

Since $T(t)$ does not vanish in $B(r)$, $r = r(M, n, \lambda)$ is the first positive zero of $T(t)$. This completes the proof. \(\square\)

**Remark.** When $M = S^m(\alpha)$ or $P^m(\alpha)$, using the properties of the Gauss hypergeometric functions, we have many results of different type which are not available for other spaces $C P^m(\alpha)$, $Q P^m(\alpha)$ and Cay $P^2(\alpha)$.

**References**


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