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PROPERTY C″ AND FUNCTION SPACES

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ABSTRACT. For a Tychonoff space X, we show that each finite product of X has property C″ iff $C_p(X)$ has countable strong fan tightness, where $C_p(X)$ is the function space on X with the topology of pointwise convergence.

1. Introduction. In this paper by a space we shall always mean a Tychonoff space. N denotes the positive integers. Unexplained notions and terminology follow [**3**]. We begin with some definitions. We denote by $C_p(X)$ the function space on a space X with the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of form $[x_1, x_2, \ldots, x_k; U_1, U_2, \ldots, U_k] = \{f \in C_p(X)\colon f(x_i) \in U_i \ i = 1, 2, \ldots, k\}$, where $k \in N$, $x_i \in X$ and each U_i is an open subset of the real-line. A space X is called strictly Fréchet if for each $x \in X$ and each countable system $\{A_n\colon n \in N\}$ of subsets of X such that $x \in \bigcap \overline{A}_n$ there exist $x_n \in A_n$ such that $x_n \to x$. The fan tightness of a space X is countable if for each $x \in X$ and each countable system $\{A_n\colon n \in N\}$ of subsets of X such that $x \in \bigcap \overline{A}_n$ there exist finite sets $B_n \subset A_n$ such that $x \in \overline{\bigcup B_n}$. The tightness of a space X is countable if for each $x \in X$ and a subset A of X such that $x \in \overline{A}$ there exists a countable subset B of A such that $x \in \overline{B}$. Obviously a strictly Fréchet space has countable fan tightness and a space with countable fan tightness has countable tightness. These 'convergence' properties in $C_p(X)$ are characterized by 'covering properties' of X [**1**, **2**, **5**]. A family of subsets $\mathscr{A}$ of a set X is said to be an ω-cover of X if for any finite subset F of X there is an $A \in \mathscr{A}$ such that $F \subset A$ [**5**]. For a sequence of subsets $\{A_n\colon n \in N\}$ of X we set $\underline{\lim} A_n = \bigcup_n \bigcap_{m \geq n} A_m$. A space X is said to have property (γ) if for each open ω-cover $\mathscr{U}$ of X there is a sequence $U_n \in \mathscr{U}$ with $\underline{\lim} U_n = X$. We note that property (γ) is finitely productive. A space X is called a Hurewicz space if for each sequence $\{\mathscr{U}_n\colon n \in N\}$ of open covers of X there exist finite $\mathscr{V}_n \subset \mathscr{U}_n$ such that $\bigcup_n \mathscr{V}_n$ covers X.

The following are known.

(1) $C_p(X)$ is strictly Fréchet iff X has property (γ) [**5**].

(2) $C_p(X)$ has countable fan tightness iff each finite product of X is a Hurewicz space [**1**].

(3) $C_p(X)$ has countable tightness iff each finite product of X is a Lindelöf space [**2**, p. 83].

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We note that among function spaces with the topology of pointwise convergence strict Fréchetness, Fréchetness, sequentiality and k-space are all equivalent [**4**, **5**]. A space X has property C$''$ if for any sequence $\{\mathscr{U}_n : n \in N\}$ of open covers of X there is a sequence $U_n \in \mathscr{U}_n$ such that $\{U_n : n \in N\}$ covers X [**6**, **9**]. Obviously a space with property C$''$ is a Hurewicz space. In [**5**] a property (δ) weaker than property (γ) is defined and it is shown that a space with property (δ) has property C$''$ [**5**, p. 159]. So, since property (γ) is finitely productive, for a space X with property (γ) each finite product of X has property C$''$. However a topological property of $C_p(X)$ corresponding to property C$''$ of X was not given in [**5**]. The purpose of this paper is to give such a property in $C_p(X)$. Property C$''$ is related to some problems of the theory of measure [**7**]. We note that the unit interval $[0, 1]$ does not satisfy property C$''$ and a space with property C$''$ is 0-dimensional. These facts are proved in the same manner as Lemma 1 and its corollary in [**5**, p. 157].

2. Results.

LEMMA. *Let X be a space.* (1) *If $\{U_n : n \in N\}$ is a countable ω-cover of X such that $U_n \neq X$ for any $n \in N$, then for each finite subset F of X $|\{n \in N : F \subset U_n\}|$ is infinite.*

(2) *The following are equivalent.*

(a) *Each finite product of X has property* C$''$.

(b) *If $\{\mathscr{U}_n : n \in N\}$ is a sequence of open ω-covers of X, then there exist $U_n \in \mathscr{U}_n$ such that $\{U_n : n \in N\}$ is an ω-cover of X.*

PROOF. (1) is easy. We prove (2). We assume the condition (a). Let $\{\mathscr{U}_n : n \in N\}$ be a sequence of open ω-covers of X. We set $\{\mathscr{U}_n\} = \{\mathscr{U}_{mn} : m, n \in N\}$. For a subset U of X we denote by $A_m(U)$ the product of m copies of U. Put $\mathscr{V}_{mn} = \{A_m(U) : U \in \mathscr{U}_{mn}\}$. Since each $\mathscr{U}_{mn}$ is an ω-cover of X, $\mathscr{V}_{mn}$ is an open cover of X^m. Since $\{\mathscr{V}_{mn} : n \in N\}$ is a sequence of open covers of X^m, by (a), there exist $U_{mn} \in \mathscr{U}_{mn}$ such that $\{A_m(U_{mn}) : n \in N\}$ covers X^m. It is easy to prove that $\{U_{mn} : m, n \in N\}$ is an ω-cover of X. Conversely we assume (b). First we show that X has property C$''$. Let $\{\mathscr{U}_n : n \in N\}$ be a sequence of open covers of X. We set $\{\mathscr{U}_n\} = \{\mathscr{U}_{mn} : m, n \in N\}$. For each m we set $\mathscr{V}_{m1} = \mathscr{U}_{m1}$, $\mathscr{V}_{m2} = \{U \cup V : U \in \mathscr{U}_{m2}, V \in \mathscr{U}_{m3}\}$, $\mathscr{V}_{m3} = \{U \cup V \cup W : U \in \mathscr{U}_{m4}, V \in \mathscr{U}_{m5}, W \in \mathscr{U}_{m6}\}, \dots$. Then it is easy to see that $\mathscr{V}_m = \bigcup_n \mathscr{V}_{mn}$ is an open ω-cover of X. By (b) there exist $V_m \in \mathscr{V}_m$ such that $\{V_m : m \in N\}$ is an ω-cover of X. This means that X has property C$''$. Next we show that the condition (b) is finitely productive. Since (b) is closed hereditary, we have only to prove that X^2 satisfies (b). Let $\{\mathscr{U}_n : n \in N\}$ be a sequence of open ω-covers of X^2. We put $\mathscr{V}_n = \{V : V \text{ open in } X, V \times V \subset U$ for some $U \in \mathscr{U}_n\}$. Since $\mathscr{U}_n$ is an ω-cover of X^2, $\mathscr{V}_n$ is also an ω-cover of X. There exist $V_n \in \mathscr{V}_n$ such that $\{V_n : n \in N\}$ is an ω-cover of X. We select $U_n \in \mathscr{U}_n$ such that $V_n \times V_n \subset U_n$. Since $\{V_n \times V_n : n \in N\}$ is an ω-cover of X^2, $\{U_n : n \in N\}$ is also an ω-cover of X^2. The proof is complete.

We say that a space X has countable strong fan tightness if for each $x \in X$ and each countable system $\{A_n : n \in N\}$ of subsets of X such that $x \in \bigcap \overline{A_n}$ there exist $x_n \in A_n$ such that $x \in \overline{\{x_n : n \in N\}}$. The proof of the following theorem is different from the proof of Theorem 4 in [**1**]. In fact the analogous proof is not available for the following theorem. We make use of the concept of an ω-cover.

THEOREM 1. *Let X be a space. The following are equivalent.*
(a) *Each finite product of X has property C″.*
(b) *$C_p(X)$ has countable strong fan tightness.*

PROOF. We assume (a). Let $f \in \bigcap_n \overline{A}_n$, where A_n is a subset of $C_p(X)$. Since $C_p(X)$ is homogeneous, we may think that f is the constant function to the zero. We set $\mathscr{U}_n = \{g^{\leftarrow}(-1/n, 1/n) \colon g \in A_n\}$ for each $n \in N$. For each $n \in N$ and each finite subset $\{x_1, x_2, \dots, x_k\}$ of X a neighborhood $[x_1, \dots, x_k; W, \dots, W]$ of f, where $W = (-1/n, 1/n)$, contains some $g \in A_n$. This means that each $\mathscr{U}_n$ is an open ω-cover of X. In case the set $M = \{n \in N \colon X \in \mathscr{U}_n\}$ is infinite, choose $g_m \in A_m$ $m \in M$ so that $g_m^{\leftarrow}(-1/m, 1/m) = X$, then $g_m \to f$. So we may assume that there exists $n \in N$ such that for each $m \geq n$ and $g \in A_m g \leftarrow (-1/m, 1/m)$ is not X. For the sequence $\{\mathscr{U}_m \colon m \geq n\}$ of open ω-covers of X, by Lemma (2), there exist $f_m \in A_m$ such that $\mathscr{U} = \{f_m^{\leftarrow}(-/m, 1/m) \colon m \geq n\}$ is an ω-cover of X. Let $[x_1, \dots, x_k; W, \dots, W]$ be any basic open neighborhood of f, where $W = (-\varepsilon, \varepsilon), \varepsilon > 0$. By lemma (1) there exists $m \geq n$ such that $\{x_1, \dots, x_k\} \subset f_m^{\leftarrow}(-1/m, 1/m)$ and $1/m < \varepsilon$. This means $f \in \overline{\{f_m \colon m \geq n\}}$. Conversely we assume (b). Let $\{\mathscr{U}_n \colon n \in N\}$ be a sequence of open ω-covers of X. We set $A_n = \{f \in C_p(X) \colon f|X - U = 0$ for some $U \in \mathscr{U}_n\}$. It is not difficult to see that each A_n is dense in $C_p(X)$ since each $\mathscr{U}_n$ is an ω-cover of X and X is Tychonoff. Let f be the constant function to 1. By the assumption there exist $f_n \in A_n$ such that $f \in \overline{\{f_n \colon n \in N\}}$. For each f_n we take $U_n \in \mathscr{U}_n$ such that $f_n|X - U_n = 0$. Set $\mathscr{U} = \{U_n \colon n \in N\}$. For each finite subset $\{x_1, \dots, x_k\}$ of X we consider the basic open neighborhood of f $[x_1, \dots, x_k; W, \dots, W]$, where $W = (0, 2)$. $[x_1, \dots, x_k; W, \dots, W]$ contains some f_n. This means $\{x_1, \dots, x_k\} \subset U_n$. Consequently $\mathscr{U}$ is an ω-cover of X. By Lemma (2) the proof is complete.

THEOREM 2. *If $C_p(X)$ has countable strong fan tightness, then so does $C_p(X)^\omega$.*

PROOF. Note that $C_p(X)^\omega$ is homeomorphic to $C_p(Y)$, where Y is the topological sum of ω copies of X. Since property C″ is closed under countable union, each finite product of Y has property C″. Thus $C_p(Y)$ has countable strong fan tightness.

Property C″ is not finitely productive. There exists a space X with property C″ such that X^2 is not normal. In [**8**, p. 216] it is noted that there exists a Hurewicz space X such that X^2 is not normal. It is easy to see that the space X has property C″.

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