ANALYTICITY OF HOMOLOGY CLASSES

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ABSTRACT. Let $W$ be a real analytic manifold and $\{\alpha\} \in H_\Sigma(W, \mathbb{Z}_2)$. We shall say that $\{\alpha\}$ is analytic if there exists a compact analytic subset $S$ of $W$, such that: $\{\alpha\} = \{\text{fundamental class of } S\}$. The purpose of this short paper is to prove

THEOREM 1. Let $W$ be a paracompact real analytic manifold; then any homology class $\{\alpha\} \in H_\Sigma(W, \mathbb{Z}_2)$ is analytic.

We remember that a similar result does not hold in the real algebraic case (see [1]).

1. Definitions and well-known facts. Let $V, W$ be two differentiable (i.e. $C^\infty$) manifolds; then, on the set $M(V,W)$ of differentiable maps $f: V \to W$, we shall consider the Whitney topology (see [2, p. 42]). In the following we shall use the known result: if $f \in M(V,W)$, then there exists a neighborhood $U$, in the $C^0$ topology, of $f$ such that any $g \in U$ is homotopic to $f$ (see [8]). By real algebraic variety we shall mean: affine real algebraic variety. A regular algebraic variety shall be called: algebraic manifold. An algebraic map is the restriction of a rational regular map.

In the following we shall need

LEMMA 1. Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^q$ be two real algebraic manifolds and $V \xrightarrow{\varphi} W$ a differentiable map. If $V$ is compact and boarding to $\varphi$, then, for any $\varepsilon > 0$, there exists an algebraic submanifold $V' \subset \mathbb{R}^{n+q}$, an analytic isomorphism $V' \xrightarrow{\pi} V$ and an algebraic map $\varphi': V' \to W$ such that

(i) $\delta(\varphi(x), \varphi' \circ \pi^{-1}(x)) < \varepsilon, x \in V,$

(ii) $\delta'(d\varphi(v), (d(\varphi' \circ \pi^{-1}))(v)) < \varepsilon$

for any tangent vector $v$, to $V$ in $x$, where $\delta, \delta'$ are two metrics on $\mathbb{R}^q$ and on the Grassmannian manifold.

PROOF. See [3].

LEMMA 2. Let $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^q$ be two real algebraic manifolds and $\varphi: V \to W$ an algebraic map. Let us suppose that $V$ is irreducible and there exists a Zariski open set $V' \subset V$ with the property: $\varphi$ is injective on $V'$. Under these hypotheses, if $T$ is the Zariski closure of $\varphi(V)$ in $W$, we have

(i) $T \supset \varphi(V)$, $\dim T = \dim V$; $T - \varphi(V)$ is contained in an algebraic set $S$, $\dim S < \dim V$,

(ii) $\{\text{fundamental class of } T\} = \varphi_* \{\text{fundamental class of } V\}$. 

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PROOF. (i) is proved in [4, Lemma 1.1]. (ii) follows from the definition of the fundamental class (see [5]) and the proof of the first part of the lemma.

Now let \( \varphi: V \to W \) be a differentiable map between differentiable manifolds. Let us denote by \( \mathcal{E}(V)_x, \mathcal{E}(W)_{\varphi(x)} \) the stalks of the sheaves of the differentiable functions on \( V, W \). We recall

**DEFINITION.** \( \varphi \) is called finite, in the point \( x \), if \( \mathcal{E}(V)_x \) is a finite \( \varphi^*(\mathcal{E}(W)_{\varphi(x)}) \) module.

We have

**LEMMA 3.** Let \( \dim V \leq \dim W \) and let us suppose that \( V \) is compact. Then the set of differentiable maps that are finite in any point is an open dense subset of \( M(V,W) \).

**PROOF.** See [6, p. 96] (see also [2, p. 169]).

**2. The proof of the theorem.** Now let us suppose that \( W \) is a compact real analytic manifold and \( \{\alpha\} \in H_p(W, \mathbb{Z}_2) \). It is known (see [7]) that we may suppose \( W \) is a real algebraic manifold. Moreover, there exists a compact differentiable manifold \( V \) and a differentiable map \( \varphi: V \to W \) such that \( \{\alpha\} = \varphi \) (fundamental class of \( V \)) (see [8]). By Lemma 1 we may suppose there exists an algebraic manifold \( \tilde{V} = V' \cup V'' \) and an algebraic map \( \tilde{\varphi}: \tilde{V} \to W \) such that:

(i) \( V' \) and \( V'' \) are diffeomorphic to \( V \),

(ii) \( \{\alpha\} = \varphi_* \) (fundamental class of \( V' \)) = \( \tilde{\varphi} \) (fundamental class of \( V'' \)),

(iii) \( \varphi|_{V'} \) is in general position with respect to \( \tilde{\varphi}(V'') \).

Moreover, by Lemma 3, we may suppose

(iv) \( \tilde{\varphi} \) is finite in any \( x \in \tilde{V} \).

From Lemma 2, we may finally assume that

(v) if \( T \) is the Zariski closure of \( \varphi(\tilde{V}) \) in \( W \), then \( T - \varphi(\tilde{V}) \) is contained in an algebraic set \( S \) such that \( \dim S < \dim V = p \).

Let now \( \tilde{\varphi}: \tilde{V} \to \tilde{W} \) be a complexification of \( \varphi \), such \( \tilde{\varphi} \) exists (see [9]), and we may suppose \( \tilde{\varphi} \) is finite in any point of \( \tilde{V} \), because the finiteness is an open condition (see [2, p. 168]). We shall suppose that \( \tilde{V} = \tilde{V}' \cup \tilde{V}'' \). The map \( \tilde{\varphi} \) is finite, hence the image of any analytic germ \( \tilde{V}_y \) is the germ of a complex analytic set of \( \tilde{W} \), see [10, p. 162].

Let us now remark that the above facts imply that:

(a) \( T = \) real part of the closure in the Zariski topology of \( \tilde{\varphi}(\tilde{V}) \),

(b) \( d\tilde{\varphi} \) has maximum rank on an open dense set of \( \tilde{V} \).

We deduce that, for any \( x \in T \), we have three disjoint analytic irreducible germs of \( T_x \):

1. germs image of \( \tilde{V}' \), of dimension \( p \),
2. germs image of \( \tilde{V}'' \), of dimension \( p \),
3. germ image of \( \tilde{\varphi}^{-1}(S) \), of dimension lower than \( p \).

This proves that \( T' = \tilde{\varphi}(V') \cup S \) is an analytic subset of \( W \) and, clearly,

\( \{\alpha\} = \{\text{fundamental class of } T'\} \).
In fact, for any $y \in T$, the germ $Y_y = \big| \bigcup_i \tilde{\varphi}(\tilde{V}_{y_i}) \big|_R$ is real analytic, where $\bigcup y_i = \tilde{\varphi}^{-1}(y) \cap \tilde{V}$, $|Z|_R = \text{real part of } Z$.

So $Y_y \cup S_y$ is real analytic and clearly $Y_y \cup S_y = T_y$. In fact, in any point, $T'$ is the union of a finite set of irreducible germs of $T$. So the theorem is proved under the hypothesis: $W$ is compact. In the general case, let us remark that we may take a representative element $\alpha$ of $\{\alpha\}$ contained in a relatively compact open set $U$ of $W$. We can now see $U$, up to analytic isomorphism, as an open set of a compact analytic manifold $Z$ (take the unique analytic structure on the double of $U$). We can now prove the analyticity of $\{\alpha\}$ in $Z$ and this implies, clearly, the analyticity of $\{\alpha\}$ in $W$. The theorem is proved.

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