PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 3, November 1988

CHARACTERIZING Ω-STABILITY FOR FLOWS IN THE PLANE

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(Communicated by Kenneth R. Meyer)

ABSTRACT. In this paper, the C^r Ω-stability for flows in the plane is characterized using the notion called "generalized recurrence" by J. Auslander [**1**].

The Ω-stability theory of vector fields on noncompact manifolds is not a simple analogue of the compact theory. The requirement that the equivalence homeomorphism, in the definition of Ω-stability, be near the identity map in the compact-open C^0 topology is essential, as it is in the definition of the global C^r structural stability of vector fields on noncompact surfaces in [**3**]. Nitecki proves there that on the plane without such requirement, structural stability does not necessarily imply that singularities be hyperbolic. The same is true for Ω-stability. He uses the idea of a composed focus of Sotomayor [**7**], that is, a singularity which is topologically a sink or a source, but it is not hyperbolic because the eigenvalues of the linearization are pure imaginary.

Let $\mathscr{X}_0^r(M)$ be the space of all C^r vector fields on a noncompact manifold M, with the C^r Whitney topology [**6**]. The vector fields which generate flows form an open subspace $\mathscr{X}^r(M) \subset \mathscr{X}_0^r(M)$. The vector field $X \in \mathscr{X}^r(M)$ determines a unique flow $\varphi\colon R \times M \to M$. For basic notions and facts about Ω-stability see [**4**].

Nitecki's statement is

1. LEMMA. *There exist C^r flows φ on $\mathbf{R}^2$ $(r \geq 4)$ which possess a composed focus and such that φ is topologically equivalent to any flow ψ C^r-near φ.*

The phrase portrait for φ is in Figure 1. The strips that contain the discs D_1 and D_{-1} are translated respectively to the right and to the left indefinitely. According to Nitecki, considering a flow ψ, C^4-near φ it is easy to see that the orbit structure of ψ outside the disc D_0 is equivalence to that of φ. Inside, ψ/D_0 is equivalent either to φ/D_0 or to φ/D_1. In the first case the equivalence takes the disc D_n to itself and in the second case, the equivalence takes the ψ-portrait in D_n to the φ-portrait in D_{n+1}. The equivalence in the second case is not near the identity, but φ and ψ are globally equivalent in both cases. For more details see [**3**]. Thus, the following definition of Ω-stability is the most appropriate for the noncompact manifolds M.

2. DEFINITION. $X \in \mathscr{X}^r(M)$ is said to be C^r *Ω-stable* if for every compact $K \subset M$ and $\varepsilon > 0$ there exists a neighborhood $\mathscr{U}$ of X in the C^r Whitney topology,

Received by the editors September 24, 1987 and, in revised form, December 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F10.
Key words and phrases. Flows, Ω-stability, generalized recurrence, prolongational limit sets, closing lemma.

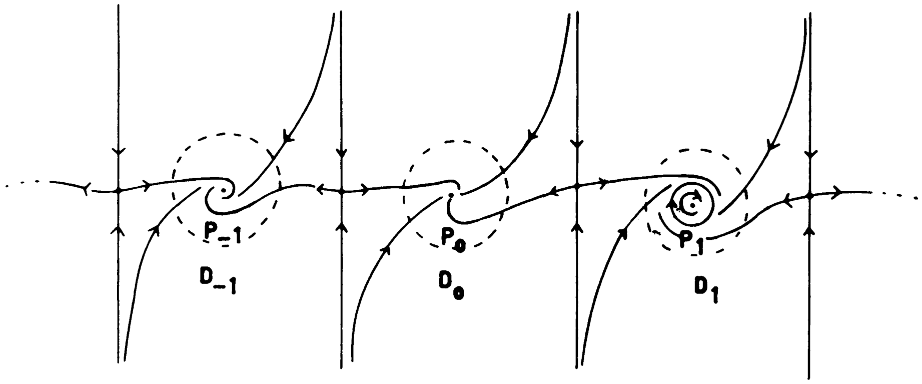

FIGURE 1

$r \geq 1$, such that for each $Y \in \mathscr{U}$ there exists a homeomorphism $h_Y : \Omega(X) \to \Omega(Y)$ which is ε-near the identity in K and takes trajectories of X into trajectories of Y.

We can characterize the Ω-stability for flows in the plane using the generalized recurrent set $R(X)$, defined by J. Auslander [**1**]. Denoting by σ_i^X the critical elements (singularities and closed orbits) of a vector field X, we state a theorem for the characterization of the Ω-stability as follows:

3. THEOREM. *$X \in \mathscr{X}^r(\mathbf{R}^2)$, $r \geq 1$ with all critical elements σ_i^X hyperbolic is Ω-stable if and only if $R(X) = \bigcup_i \sigma_i^X$.*

We now define $R(X)$ and for this we need a few other definitions.

4. DEFINITION. For each $X \in \mathscr{X}^r(\mathbf{R}^2)$ and $x \in \mathbf{R}^2$ we define the *first prolongational limit set* by

$$J^+(x) = \{y \in \mathbf{R}^2 : \exists x_n \to x,\ t_n \to \infty \ni \varphi_{t_n}(x_n) \to y\}.$$

5. DEFINITION. For a subset $S \subset \mathbf{R}^2$, we define

$$J^+(S) = \bigcup_{x \in S} J^+(x).$$

6. DEFINITION. For each ordinal number α, $X \in \mathscr{X}^r(\mathbf{R}^2)$ and $x \in \mathbf{R}^2$ we call $J_\alpha(x)$ the prolongational limit set of order α, defined by transfinite induction as follows:

(1) $J_1(x) = J^+(x)$.

(2) Suppose that for all $\beta < \alpha$, $J_\beta(x)$ is defined.

(i) if α is a successor ordinal number, we set $J_\alpha(x) = J_1(J_{\alpha-1}(x))$;

(ii) if α is a limit ordinal number, we set

$$J_\alpha(x) = \{y \in \mathbf{R}^2 : \exists x_n \to x,\ y_n \to y \text{ and ordinals } \beta_n < \alpha \text{ with } y_n \in J_{\beta_n}(x_n)\}.$$

7. DEFINITION. *The generalized or prolongational recurrent set* $R(X)$ of X, also called the Auslander recurrent set, is defined by

$$R(X) = \{x \in \mathbf{R}^2 : x \in J_\alpha(x) \text{ for some ordinal number } \alpha\}.$$

8. DEFINITION. A point $p \in R(X)$ which is not a periodic point is called *prolongationally recurrent point.*

We use here a weaker version of the Closing Lemma for the generalized recurrent set in [**5**]:

9. CLOSING LEMMA FOR $R(X)$. *Suppose* $X \in \mathscr{X}^r(\mathbf{R}^2)$ *has only hyperbolic singularities. Given* $p \in R(X)$, *a prolongationally recurrent point, and a neighborhood* $\mathscr{U}$ *of* X *in* $\mathscr{X}^r(\mathbf{R}^2)$ *then there exists* $Y \in \mathscr{U}$ *with a closed orbit through* p.

We prove the necessary condition of Theorem 3 in the following.

10. PROPOSITION. *If* $X \in \mathscr{X}^r(\mathbf{R}^2)$ *is* Ω*-stable, then* $R(X) = \Omega(X) = \bigcup_i \sigma_i^X$.

PROOF. Suppose $R(X) \neq \bigcup_i \sigma_i^X$. Consider $p \in R(X) - \bigcup_i \sigma_i^X$. Choose $\varepsilon > 0$ and K a compact disc with radius greater than ε and center in p and such that no closed orbit of X intersects K. Since X is Ω-stable, there exists a neighborhood $\mathscr{U}$ of X such that for any $Y \in \mathscr{U}$ there exists a homeomorphism h_Y between $\Omega(X)$ and $\Omega(Y)$, which is a ε-homeomorphism on K. By the Closing Lemma for $R(X)$, there exists a vector field $Y \in \mathscr{U}$, which has a closed orbit γ_Y through p. Since γ_Y crosses K, by the ε-homeomorphism h_Y in K, X has a closed orbit γ_X, ε-near γ_Y, crossing K. This is a contradiction. Then $R(X) = \bigcup_i \sigma_i^X$. Since $\bigcup_i \sigma_i^X \subset \Omega(X) \subset R(X)$, we have $R(X) = \Omega(X) = \bigcup_i \sigma_i^X$.

The condition $R(x) = \bigcup_i \sigma_i^X$ is also a sufficient condition for Ω-stability for vector fields in $\mathbf{R}^2$ and this was proved by Fopke Klok in [**2**]. Although Fopke Klok does not take into account that in the definition of Ω-stability the equivalence homeomorphism should be C^0-near the identity map on $\mathbf{R}^2$ (with respect to the compact open topology), a simple modification of this proof makes it work well for the case when Ω-stability is defined as in (2). This happens because we have a finite number of critical elements of X in each compact $\mathscr{C} \subset \mathbf{R}^2$ by the following proposition.

11. PROPOSITION. *If* $X \in \mathscr{X}^r(\mathbf{R}^2)$ *has all critical elements* σ_i^X *hyperbolic and* $R(X) = \bigcup_i \sigma_i^X$, *then for each compact set* $\mathscr{C} \subset \mathbf{R}^2$ *there is only a finite number of* σ_i^X *with* $\sigma_i^X \cap \mathscr{C} \neq \varnothing$. *For the proof see* [**2**].

This enables us to choose, for vector fields as above, disjoint and sufficiently small neighborhoods of the critical elements σ_i^X, in order to prove the existence of a neighborhood $\mathscr{U}(X)$ such that the equivalence homeomorphism between $\Omega(X)$ and $\Omega(Y)$ be C^0-near the identity in given compacts sets, for $Y \in \mathscr{U}(X)$. Thus the Ω-stability for vector fields in the plane is characterized by Theorem 3.

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