CHARACTERIZING $\Omega$-STABILITY
FOR FLOWS IN THE PLANE
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ABSTRACT. In this paper, the $C^r$ $\Omega$-stability for flows in the plane is characterized using the notion called "generalized recurrence" by J. Auslander [1].

The $\Omega$-stability theory of vector fields on noncompact manifolds is not a simple analogue of the compact theory. The requirement that the equivalence homemorphism, in the definition of $\Omega$-stability, be near the identity map in the compact-open $C^0$ topology is essential, as it is in the definition of the global $C^r$ structural stability of vector fields on noncompact surfaces in [3]. Nitecki proves there that on the plane without such requirement, structural stability does not necessarily imply that singularities be hyperbolic. The same is true for $\Omega$-stability. He uses the idea of a composed focus of Sotomayor [7], that is, a singularity which is topologically a sink or a source, but it is not hyperbolic because the eigenvalues of the linearization are pure imaginary.

Let $\mathcal{X}(M)$ be the space of all $C^r$ vector fields on a noncompact manifold $M$, with the $C^r$ Whitney topology [6]. The vector fields which generate flows form an open subspace $\mathcal{X}(M) \subset \mathcal{X}(M)$. The vector field $X \in \mathcal{X}(M)$ determines a unique flow $\varphi: \mathbb{R} \times M \to M$. For basic notions and facts about $\Omega$-stability see [4].

Nitecki's statement is

1. LEMMA. There exist $C^r$ flows $\varphi$ on $\mathbb{R}^2$ ($r \geq 4$) which possess a composed focus and such that $\varphi$ is topologically equivalent to any flow $\psi$ $C^r$-near $\varphi$.

The phrase portrait for $\varphi$ is in Figure 1. The strips that contain the discs $D_1$ and $D_{-1}$ are translated respectively to the right and to the left indefinitely. According to Nitecki, considering a flow $\psi$, $C^4$-near $\varphi$ it is easy to see that the orbit structure of $\psi$ outside the disc $D_0$ is equivalence to that of $\varphi$. Inside, $\psi/D_0$ is equivalent either to $\varphi/D_0$ or to $\varphi/D_1$. In the first case the equivalence takes the disc $D_n$ to itself and in the second case, the equivalence takes the $\psi$-portrait in $D_n$ to the $\varphi$-portrait in $D_{n+1}$. The equivalence in the second case is not near the identity, but $\varphi$ and $\psi$ are globally equivalent in both cases. For more details see [3]. Thus, the following definition of $\Omega$-stability is the most appropriate for the noncompact manifolds $M$.

2. DEFINITION. $X \in \mathcal{X}(M)$ is said to be $C^r$ $\Omega$-stable if for every compact $K \subset M$ and $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of $X$ in the $C^r$ Whitney topology,
r ≥ 1, such that for each Y ∈ ℳ there exists a homeomorphism h_Y : Ω(X) → Ω(Y) which is ε-near the identity in K and takes trajectories of X into trajectories of Y.

We can characterize the Ω-stability for flows in the plane using the generalized recurrent set R(X), defined by J. Auslander [1]. Denoting by σ_i^X the critical elements (singularities and closed orbits) of a vector field X, we state a theorem for the characterization of the Ω-stability as follows:

3. THEOREM. X ∈ ℳ^r(R^2), r ≥ 1 with all critical elements σ_i^X hyperbolic is Ω-stable if and only if R(X) = ∪_i σ_i^X.

We now define R(X) and for this we need a few other definitions.

4. DEFINITION. For each X ∈ ℳ^r(R^2) and x ∈ R^2 we define the first prolongational limit set by

\[ J^+(x) = \{ y ∈ R^2 : \exists x_n → x, t_n → ∞ \exists \varphi_{t_n}(x_n) → y \}. \]

5. DEFINITION. For a subset S ⊂ R^2, we define

\[ J^+(S) = \bigcup_{x ∈ S} J^+(x). \]

6. DEFINITION. For each ordinal number α, X ∈ ℳ^r(R^2) and x ∈ R^2 we call J_α(x) the prolongational limit set of order α, defined by transfinite induction as follows:

(1) J_1(x) = J^+(x).

(2) Suppose that for all β < α, J_β(x) is defined.

(i) if α is a successor ordinal number, we set J_α(x) = J_1(J_{α-1}(x));

(ii) if α is a limit ordinal number, we set

\[ J_α(x) = \{ y ∈ R^2 : \exists x_n → x, y_n → y \text{ and ordinals } β_n < α \text{ with } y_n ∈ J_{β_n}(x_n) \}. \]

7. DEFINITION. The generalized or prolongational recurrent set R(X) of X, also called the Auslander recurrent set, is defined by

\[ R(X) = \{ x ∈ R^2 : x ∈ J_α(x) \text{ for some ordinal number } α \}. \]

8. DEFINITION. A point p ∈ R(X) which is not a periodic point is called prolongationally recurrent point.
We use here a weaker version of the Closing Lemma for the generalized recurrent set in [5]:

9. CLOSING LEMMA FOR $R(X)$. Suppose $X \in \mathcal{D}^r(\mathbb{R}^2)$ has only hyperbolic singularities. Given $p \in R(X)$, a prolongationally recurrent point, and a neighborhood $\mathcal{U}$ of $X$ in $\mathcal{D}^r(\mathbb{R}^2)$ then there exists $Y \in \mathcal{U}$ with a closed orbit through $p$.

We prove the necessary condition of Theorem 3 in the following.

10. PROPOSITION. If $X \in \mathcal{D}^r(\mathbb{R}^2)$ is $\Omega$-stable, then $R(X) = \Omega(X) = \bigcup_i \sigma_i^X$.

PROOF. Suppose $R(X) \neq \bigcup_i \sigma_i^X$. Consider $p \in R(X) - \bigcup_i \sigma_i^X$. Choose $\epsilon > 0$ and $K$ a compact disc with radius greater than $\epsilon$ and center in $p$ and such that no closed orbit of $X$ intersects $K$. Since $X$ is $\Omega$-stable, there exists a neighborhood $\mathcal{U}$ of $X$ such that for any $Y \in \mathcal{U}$ there exists a homeomorphism $h_Y$ between $\Omega(X)$ and $\Omega(Y)$, which is a $\epsilon$-homeomorphism on $K$. By the Closing Lemma for $R(X)$, there exists a vector field $Y \in \mathcal{U}$, which has a closed orbit $\gamma_Y$ through $p$. Since $\gamma_Y$ crosses $K$, by the $\epsilon$-homeomorphism $h_Y$ in $K$, $X$ has a closed orbit $\gamma_X$, $\epsilon$-near $\gamma_Y$, crossing $K$. This is a contradiction. Then $R(X) = \bigcup_i \sigma_i^X$. Since $\bigcup_i \sigma_i^X \subset \Omega(X) \subset R(X)$, we have $R(X) = \Omega(X) = \bigcup_i \sigma_i^X$.

The condition $R(x) = \bigcup_i \sigma_i^X$ is also a sufficient condition for $\Omega$-stability for vector fields in $\mathbb{R}^2$ and this was proved by Fopke Klok in [2]. Although Fopke Klok does not take into account that in the definition of $\Omega$-stability the equivalence homeomorphism should be $C^0$-near the identity map on $\mathbb{R}^2$ (with respect to the compact open topology), a simple modification of this proof makes it work well for the case when $\Omega$-stability is defined as in (2). This happens because we have a finite number of critical elements of $X$ in each compact $\mathcal{C} \subset \mathbb{R}^2$ by the following proposition.

11. PROPOSITION. If $X \in \mathcal{D}^r(\mathbb{R}^2)$ has all critical elements $\sigma_i^X$ hyperbolic and $R(X) = \bigcup_i \sigma_i^X$, then for each compact set $\mathcal{C} \subset \mathbb{R}^2$ there is only a finite number of $\sigma_i^X$ with $\sigma_i^X \cap \mathcal{C} \neq \emptyset$. For the proof see [2].

This enables us to choose, for vector fields as above, disjoint and sufficiently small neighborhoods of the critical elements $\sigma_i^X$, in order to prove the existence of a neighborhood $\mathcal{U}(X)$ such that the equivalence homeomorphism between $\Omega(X)$ and $\Omega(Y)$ be $C^0$-near the identity in given compacts sets, for $Y \in \mathcal{U}(X)$. Thus the $\Omega$-stability for vector fields in the plane is characterized by Theorem 3.

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