COMPACTIFICATIONS OF COUNTABLE-DIMENSIONAL
AND STRONGLY COUNTABLE-DIMENSIONAL SPACES

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ABSTRACT. Simple proofs of theorems on existence of compactifications of
countable-dimensional and strongly countable-dimensional spaces are given.

In this note we present simple proofs of the following two theorems (for the
terminology see [2 and 3]):

THEOREM 1. Every countable-dimensional completely metrizable separable
space has a countable-dimensional metrizable compactification.

THEOREM 2. Every strongly countable-dimensional completely metrizable sep-
arable space has a strongly countable-dimensional metrizable compactification.

Let us recall that a completely regular space $X$ is (strongly) countable-dimen-
sional if $X$ can be represented as a union of a sequence $X_1, X_2, \ldots$ of (closed)
subspaces with covering dimension $\dim X_i < \infty$ (see [2] and [3]).

Theorem 1 was announced by Hurewicz in [5] and proved by Lelek in [8]. The-
orem 2 was established by Schurle in [9].

Theorem 1 follows immediately from the Lemma we state below. The original
proof of the Lemma sketched in [8] was based on the technique of Kuratowski’s
$\kappa$-mappings to infinite polyhedra; cf. [7, §28, IX]; a short and elegant proof can be
found in [4], it uses, however, some deep facts from infinite-dimensional topology.
We give here an elementary proof.

LEMMA. For every completely metrizable separable space $X$ there exists an em-
bedding $h: X \to Z$ into a compact metrizable space $Z$ such that $Z \setminus h(X)$ is a count-
able union of finite-dimensional compact spaces.

PROOF. One can consider $X$ as a subspace of a compact metrizable space $Y$,
and $X$ is the intersection of countably many open subsets $U_1 \supset U_2 \supset \cdots$ of $Y$ (see
[2, Theorem 4.3.24]). Let $\rho$ be an arbitrary metric on the space $Y$ that is bounded
by 1. Since $\rho$ is totally bounded, for $i = 1, 2, \ldots$ the set $U_i$ can be represented as
the union of finitely many open sets $U_{ij} \subset Y$, where $j \in J_i$, with diameter less than
$1/i$; let $f_{ij}: Y \to I$ be the continuous function defined by $f_{ij}(y) = \rho(y, Y \setminus U_{ij})$ for
$i = 1, 2, \ldots$ and $j \in J_i$. Denote by $f$ the diagonal of the functions $f_{ij}$; clearly,
$f: Y \to I^{\aleph_0}$ and the restriction $h = f|X$ separates points and closed sets, and
thus is an embedding of $X$ into $Z = f(X)$. As one easily sees, $Y \setminus X = f^{-1}(K_\omega)$,
where $K_\omega$ is the subspace of $I^{\aleph_0}$ consisting of all points that have only finitely

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many nonzero coordinates; obviously, $K_\omega$ is the union of countably many finite-dimensional cubes. Now, $Z \setminus h(X) \subset K_\omega$, and being an $F_\sigma$-set in $K_\omega$ is a countable union of finite-dimensional compact spaces. □

In our proof of Theorem 2 we use a very special case of the separation theorem for Borel sets (see [7, §30, VII]), viz. the fact that for every pair $A, B$ of disjoint $G_\delta$-sets in a metrizable space $M$ there exists a set $C$ which is both an $F_\sigma$ and a $G_\delta$-set such that $A \subset C \subset M \setminus B$. This can be easily verified: if $A = \bigcap_{i=1}^{\infty} G_i$ and $B = \bigcup_{i=1}^{\infty} H_i$, where the sets $M = G_1 \supset G_2 \supset \cdots$ and $M = H_1 \supset H_2 \supset \cdots$ are open in $M$, the $F_\sigma$-set $C = \bigcup_{i=1}^{\infty} (G_i \setminus H_i)$ has the required properties, since $M \setminus C = \bigcup_{i=1}^{\infty} (H_i \setminus G_{i+1})$ is an $F_\sigma$-set, too.

**Proof of Theorem 2.** Consider a strongly countable-dimensional completely metrizable space $X$; let $X = \bigcup_{i=1}^{\infty} F_i$, where $F_i$ are finite-dimensional closed subsets of $X$. By a theorem of Hurewicz [6]; for a proof, see [7, §45, VII]), there exists a metrizable compactification $M$ of the space $X$ such that for $i = 1, 2, \ldots$ we have $\dim \overline{F_i} = \dim F_i$, where the closure is taken in $M$. The sets $A = X$ and $B = M \setminus \bigcup_{i=1}^{\infty} F_i$ are disjoint $G_\delta$-sets in $M$, so that there exists a set $Y$ which is both an $F_\sigma$ and a $G_\delta$-set such that $X \subset Y \subset \bigcup_{i=1}^{\infty} F_i$. The space $Y$ is a countable union of finite-dimensional compact spaces; it is also completely metrizable, so that by the Lemma it has a metrizable compactification $Z$ obtained by adjoining to $Y$ countably many finite-dimensional compact spaces. Obviously, $Z$ is a strongly countable-dimensional compactification of the space $X$. □

To conclude, let us observe that our proof of the Lemma yields a more general result:

**Proposition 1.** For every completely regular space $X$ that has the following property:

$$(*) \quad \text{X is the intersection of countably many open subsets } U_1 \supset U_2 \supset \cdots \text{ of a compact space } Y \text{ and for } i = 1, 2, \ldots \text{ the set } U_i \text{ can be represented as the union of a point-finite family } \{U_{is}\}_{s \in S_i} \text{ of functionally open subsets of } Y \text{ in such a way that for every pair } x_1, x_2 \text{ of distinct points of } X \text{ there exists an } i \text{ and an } s \in S_i \text{ such that the set } U_{is} \text{ contains exactly one of the points } x_1, x_2,$$

there exists an embedding $h: X \to Z$ into a compact space $Z$ such that $Z \setminus h(X)$ is a countable union of finite-dimensional compact spaces.

**Proof.** Let $f_{is}: Y \to I$ be a continuous function such that $f_{is}^{-1}(0) = Y \setminus U_{is}$ for $i = 1, 2, \ldots$ and $s \in S_i$. The diagonal $f$ of the functions $f_{is}$ is a continuous mapping of $Y$ into a Tychonoff cube $I^m$, the restriction $h = f|X$ is a one-to-one mapping, and $Y \setminus X = f^{-1}(K)$, where $K$ is the subspace of $I^m$ consisting of all points that have only finitely many nonzero coordinates. Since $h$ is the restriction of the perfect mapping $f$ to the set $X = f^{-1}(I^m \setminus K)$, it is perfect itself (see [2, Proposition 3.7.4]), and being one-to-one is an embedding of $X$ into $Z = f(X)$. The remainder $Z \setminus h(X)$ is an $F_\sigma$-set in $Z$ (see [2, Theorem 3.9.1]) and is contained in $K$, so that to conclude the proof it suffices to show that $K$ is the union of countably many finite-dimensional compact spaces.

Clearly, $K = \bigcup_{n=0}^{\infty} K_n$, where $K_n$ consists of all points in $K$ that have at most $n$ nonzero coordinates. The subspaces $K_n$ being compact, it suffices to show that
dim $K_n \leq n$ for $n = 0, 1, \ldots$. This is done by induction. Since $K_0$ consists of one point, dim $K_0 = 0$. Assume that dim $K_n \leq n$. One easily checks that each point in $K_{n+1}\setminus K_n$ has a neighborhood in $K_{n+1}$ homeomorphic to the cube $(a, 1]^{n+1}$, where $0 < a < 1$; thus, for each closed set $F \subseteq K_{n+1}$ disjoint from $K_n$ we have $\dim F \leq n + 1$ by the sum theorem, so that $\dim K_{n+1} \leq n + 1$ (see [2, Theorem 7.2.1 and Problem 7.4.17]). □

One easily checks that the class of completely regular spaces that have property (*) is hereditary with respect to closed subspaces and countably multiplicative. Since every completely metrizable space is for some $m \geq \aleph_0$ homeomorphic to a closed subspace of $[J(m)]^{\aleph_0}$, where $J(m)$ is the hedgehog space of spininess $m$ (see [2, Exercise 4.4B]), to show that all completely metrizable spaces have property (*), it suffices to find an appropriate compact space containing $J(m)$. Such a space is, however, easily defined (cf. the proof of Theorem 14 in [1]): if $S$ is the set of cardinality $m$ used in the construction of $J(m)$, as described in [2, Example 4.1.5], then the quotient space $Y = (I \times \omega S)/(\{0\} \times \omega S)$, where $\omega S$ is the one-point compactification of the discrete space $S$, has all the required properties.

Thus, one obtains the following proposition that answers a question in [3] (the first part of Problem 5.4):

**Proposition 2.** Every countable-dimensional completely metrizable space has a countable-dimensional compactification. □

**References**


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