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AN EMPTY CLASS OF NONMETRIC SPACES

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ABSTRACT. Let CSSM be the class of compact nonmetrizable spaces in which every subspace of cardinality at most ω_1 is metrizable. We show that CSSM is empty.

For the purposes of this article only let us call a space an SSM space (small subspaces metrizable) if it is not metrizable but it is regular and all of its subspaces of cardinality at most ω_1 are metrizable. A CSSM space is a compact SSM space. If X is a CSSM space, then X is first countable [**HJ**]. Therefore under the continuum hypothesis (CH) there are no CSSM spaces because, of course, a compact first countable space has cardinality at most c. This was first observed by Juhasz who then asked if the CH assumption could be removed [**J**]. It was shown in [**D**] that it is consistent with (and independent of) $\neg$ CH that there are no Lindelöf, countably compact or even ω_1-compact first countable SSM spaces. In this article we show that there simply are never any CSSM spaces.

There is however an easy example, under MA+$\neg$CH, of a Lindelöf first countable SSM space. I do not know if a Lindelof SSM space is necessarily first countable.

EXAMPLE 1. Recall that the Alexandroff double topology on $I \times 2$ (where I is the unit interval) is obtained by declaring $I \times \{1\}$ to be open and discrete while a basic open neighbourhood of a point $(r, 0)$ is $U \times 2 - \{(r, 1)\}$ where $r \in U$ is open in I. If $A \subset I$ is any uncountable set containing no uncountable closed set, then $X = (I - A \times \{0\}) \cup (A \times \{1\})$ is a Lindelof non metrizable subspace of the Alexandroff double.

Furthermore, if MA(ω_1) is assumed then X is an SSM space since $A \times \{1\}$ will be an F_σ-set in any subspace of X of cardinality ω_1 (see [**M**]).

One might hope to modify the Alexandroff double somehow to obtain a CSSM space. In fact if X were a CSSM space then X would contain an uncountable discrete subset D; hencc $\operatorname{cl} D$ would itself be a CSSM space. (To see that X would contain such a D see 2(ii).)

2. PROPOSITION. *If X is a SSM space then*:

(i) *each separable subspace is metrizable and*

(ii) *X contains an uncountable discrete subset and*

(iii) *if X is, in addition, compact then X contains an uncountable discrete set D whose closure is a CSSM space.*

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PROOF. (i) is essentially due to Hajnal and Juhasz [**HJ**]. They prove that a space has countable weight if all of its subspaces of cardinality at most ω_1 do. Now (i) follows since if $K \subset X$ is separable, each subspace of size ω_1 is contained in a separable metrizable space (since X is SSM and K is separable).

Of course (ii) is obvious since, by (i), X is not separable and therefore X has a non separable subspace of cardinality ω_1. This subspace is metrizable, hence not ccc. For (iii), let D be given by (ii) and note that $\operatorname{cl} D$ is not metrizable and is therefore a CSSM space.

3. **CSSM is empty.** Suppose that X is a compact space containing the discrete space ω_1 as a dense subspace. For each $x \in X$ fix a countable neighbourhood base $\{U(x,n) : n < \omega\}$. Let $\mathscr{U} = \{\{U(x,n) : n < \omega\} : x \in X\}$.

For each $\lambda < \omega_1$ let $\mathscr{C}\lambda = \bigcap\{\operatorname{cl}(\lambda - \alpha) : \alpha < \lambda\}$.

We shall define by induction on $\gamma < \omega_1$ a continuous increasing sequence $\{M(\gamma) : \gamma < \omega_1\}$ of countable elementary submodels of some sufficiently large $H(\theta)$ and for each $\lambda(\gamma) = M(\gamma) \cap \omega_1$ we will choose $x(\gamma)$ in $\mathscr{C}\lambda(\gamma)$ as follows.

Suppose $\gamma < \omega_1$ and $\{M(\rho) : \rho < \gamma\}$, $\{x(\rho) : \rho < \gamma\}$ have been chosen so that $\{X, \mathscr{U}\} \in M(\rho)$ and $\{x(\rho), M(\rho)\} \in M(\rho + 1)$ for each $\rho < \gamma$.

In case γ is a limit, let $M(\gamma) = \bigcup\{M(\rho) : \rho < \gamma\}$. If $\gamma = \rho + 1$ let $M(\gamma)$ be any countable elementary submodel of $H(\theta)$ containing $\{M(\rho), x(\rho)\}$.

The following fact is probably of some interest by itself and it provides the basis for the whole proof.

FACT 1. If M is a countable elementary submodel of $H(\theta)$ such that $X, \mathscr{U}$ are in M, $\lambda = M \cap \omega_1$, and if $F \in [X \cap M]^{<\omega}$ and $p \colon F \to \omega$ are such that $\mathscr{C}\lambda \subset \bigcup\{U(x, p(x)) : x \in F\}$ then

$$\text{for some } \beta < \lambda \qquad [\beta, \omega_1) \subset \bigcup\{U(x, p(x)) : x \in F\}.$$

PROOF. By definition of $\mathscr{C}\lambda$, each sequence cofinal in λ is eventually in $\bigcup\{U(x, p(x)) : x \in F\}$, hence there is some $\beta < \lambda$ such that $[\beta, \lambda) \subset \bigcup\{U(x, p(x)) : x \in F\}$. But since F, p and ω_1 are all in M, we have that M is a model of $[\beta, \omega_1) \subset \bigcup\{U(x, p(x)) : x \in F\}$. Now the fact follows since M is an elementary submodel.

NOTATION. For $\rho < \gamma$ let $U(\rho, n) = U(x(\rho), n)$.

FACT 2. There is an $x(\gamma) \in \mathscr{C}\lambda(\gamma)$ such that for any $\rho < \gamma$ and $n < \omega$, $x(\gamma) \in U(\rho, n) \to \lambda(\gamma) \in U(\rho, n)$.

PROOF. If not we could find for each $x \in \mathscr{C}\lambda(\gamma)$ a pair $t(x) \in \gamma \times \omega$ such that $x \in U(t(x))$ and $\lambda(\gamma) \notin U(t(x))$. Since $\mathscr{C}\lambda(\gamma)$ is compact we find $F \in [\gamma]^{<\omega}$ such that $\bigcup\{U(t(x)) : x \in F\} \supset \mathscr{C}\lambda(\gamma)$. However, this contradicts Fact 1 since $\lambda(\gamma) \notin U(t(x))$ for $x \in F$.

Therefore we have defined a cub $\{\lambda(\gamma) : \gamma < \omega_1\}$, a sequence $\{x(\gamma) : \gamma < \omega_1\}$ with $x(\gamma) \in \mathscr{C}\lambda(\gamma)$ and a sequence of neighbourhood bases $\{U(\gamma, n) : \gamma < \omega_1, n < \omega\}$ so that $\rho < \gamma$ and $x(\gamma) \in U(\rho, n) \to \lambda(\gamma) \in U(\rho, n)$.

FACT 3. $\omega_1 \cup \{x(\gamma) : \gamma < \omega_1\}$ is not metrizable.

PROOF. Assume that it is metrizable. Recall that each open subset of a metric space is an F_σ. Therefore there must be a stationary set $S \subset \{\lambda(\gamma) : \gamma < \omega_1\}$ such that $\operatorname{cl} S \cap \{x(\gamma) : \gamma < \omega_1\} = \varnothing$ since ω_1 is open. For each $\lambda(\gamma) \in S$ choose $n(\gamma) < \omega$ so that $U(x(\gamma), n(\gamma)) \cap S = \varnothing$. It follows that if $\lambda(\gamma) < \lambda(\rho)$ are both in S then $x(\rho) \notin U(x(\gamma), n(\gamma))$ since $\lambda(\rho) \notin U(x(\gamma), n(\gamma))$. Let, for $n \in \omega$, $\mathscr{V}_n$ be a locally finite family of open subsets of $\omega_1 \cup \{x(\gamma) : \gamma < \omega_1\}$ such that $\bigcup\{\mathscr{V}_n : n \in \omega\}$ is a base (recall that each metric space has a σ-locally finite base). For each $\lambda(\gamma) \in S$, there is an m_γ and a $V_\gamma \in \mathscr{V}_{m_\gamma}$ such that $x(\gamma) \in V_\gamma \subset U(x(\gamma), n(\gamma))$. There is an $m \in \omega$ such that $S' = \{\lambda(\gamma) \in S \colon m_\gamma = m\}$ is stationary. Since $\lambda(\gamma) < \lambda(\rho)$ both in S' implies $x(\rho) \notin V_\gamma$, these sets are all distinct (i.e. $V_\gamma \neq V_\rho$). However, for each $\lambda(\gamma) \in S'$, $V_\gamma \cap [0, \lambda(\gamma)) \neq \varnothing$ since $x(\gamma)$ is a limit point of $[0, \lambda(\gamma))$. Now a pressing down argument gives that the family $\{V_\gamma \colon \lambda(\gamma) \in S'\}$ is not point-finite, contradicting that $\mathscr{V}_m$ is locally finite.

FACT 4. X is not SSM.

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