A REAL VALUED HOMOMORPHISM ON ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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(Communicated by R. Daniel Mauldin)

ABSTRACT. In this paper we prove that, for every homomorphism \( A \) on \( C^k(E) \), there exists \( x \in E \) such that \( A(f) = f(x) \) for \( f \in C^k(E) \). Here \( C^k(E) \) (\( k = 1, 2, \ldots, \infty \)) denotes the algebra of all \( k \)-times differentiable real functions on a real and separable Banach space \( E \).

Introduction. Let \( C^k(E) \) (\( k = 1, 2, \ldots, \infty \)) be the algebra of all \( k \)-times differentiable function \( f: E \to \mathbb{R} \) defined on a real Banach space \( E \). Since \( C^k(E) \) is an algebra of functions it makes sense to ask if, to each nonzero homomorphism \( A: C^k(E) \to \mathbb{R} \), there corresponds \( x \in E \) such that \( A(f) = f(x) \) for all \( f \in C^k(E) \), in other words, if \( A \) is an evaluation.

If \( k = 0 \), the answer is affirmative if and only if \( E \) is realcompact. It turns out, since \( E \) is a metric space, that it is realcompact if and only if \( \text{card}(E) \) is nonmeasurable (cf. Gillman and Jerison [1, pp. 226–232]).

On the other hand Jaramillo [4] defining two topologies \( \tau^k_\omega \) and \( \tau^k_\delta \) on \( C^k(E) \), have proved that, if \( \tau^k_\omega = \tau^k_\delta \), then every homomorphism on \( C^k(E) \) is an evaluation. They have also proved that \( \tau^k_\omega = \tau^k_\delta \) if \( E \) has nonmeasurable cardinal and \( E \) admits \( B^k \) partitions of unity. To see what kind of Banach spaces admits \( B^k \) partitions of unity, see Wells [6] and Sundaresan and Swaminathan [5].

In this paper we generalize the result of Jaramillo to all separable Banach spaces. We give a direct proof without using any topology on \( C^k(E) \) nor partitions of unity on \( E \).

1. Associated filter to a real homomorphism. In the following, \( A: C^k(E) \to \mathbb{R} \) denotes a nonzero homomorphism. For \( \alpha \in \mathbb{R} \), let \( \hat{\alpha} \) be the constant function that associates the value \( \alpha \) to each element \( x \in E \). We claim that \( A(\hat{\alpha}) = \alpha \) for every \( \alpha \in \mathbb{R} \). Indeed, \( A(\hat{1}) \neq 0 \) because \( A \neq 0 \). Then \( B: \mathbb{R} \to \mathbb{R} \) defined by \( B(\alpha) = A(\hat{\alpha}) \) is a nonzero homomorphism from \( \mathbb{R} \) into \( \mathbb{R} \). Therefore \( B \) is the identity homomorphism, that is \( A(\hat{\alpha}) = \alpha \) for every \( \alpha \in \mathbb{R} \).

Proposition 1. Let \( A: C^k(E) \to \mathbb{R} \) be a nonzero homomorphism. For every \( f \in C^k(E) \), the set \( \{x \in E: A(f) = f(x)\} \) is nonempty.

Proof. Suppose \( \{x \in E: A(f) = f(x)\} = \emptyset \). Then \( f - A(f) \) is a real function, which does not take the value 0. So \( (f - A(f))^{-1} \in C^k(E) \) and we see that

\[ 1 = A(\hat{1}) = A((f - A(f))(f - A(f))^{-1}) = A(f - A(f))A((f - A(f))^{-1}) \]

which contradicts \( A(f - A(f)) = A(f) - A(f) = 0 \).

Received by the editors October 6, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 46J15, 46J20; Secondary 41A65.
**Proposition 2.** Let \( A : C^k(E) \to \mathbb{R} \) be a nonzero homomorphism. Denote by \( S(f_1, \ldots, f_n) \) the set \( \{ x \in E : A(f_i) = f_i(x), \ i = 1, 2, \ldots, n \} \). The collection \( \mathcal{S} \) of all the sets \( S(f_1, \ldots, f_n) \) where \( f_1, \ldots, f_n \in C^k(E) \) is a filter basis on \( E \).

**Proof.** Since \( S(f_1, \ldots, f_n) \cap S(g_1, \ldots, g_m) = S(f_1, \ldots, f_n, g_1, \ldots, g_m) \), it suffices to prove that \( S(f_1, \ldots, f_n) \neq \emptyset \) for every \( f_1, \ldots, f_n \in C^k(E) \).

Consider the function \( g = \sum_{i=1}^{n} (f_i - \overline{A(f_i)})^2 \in C^k(E) \). By Proposition 1, there exists \( x \in E \) such that \( A(g) = g(x) \). From the definition of \( g \) we get \( A(g) = 0 \), so that \( \sum (f_i(x) - A(f_i))^2 = 0 \) which forces \( f_i(x) = A(f_i) \) for every \( i = 1, 2, \ldots, n \).

**Definition 3.** Let \( A : C^k(E) \to \mathbb{R} \) be a nonzero homomorphism. We will say that the filter \( \mathcal{F} \) generated by the filter basis \( \mathcal{S} \), from Proposition 2, is the filter associated to \( A \).

**Proposition 4.** Let \( A : C^k(\mathbb{R}^n) \to \mathbb{R} \) be a nonzero homomorphism. Then there exists \( x \in \mathbb{R}^n \) such that for every \( f \in C^k(\mathbb{R}^n) \), \( A(f) = f(x) \).

**Proof.** Let \( f \in C^k(\mathbb{R}^n) \) be the function defined by \( f(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n \). The filter \( \mathcal{F} \) associated to \( A \) contains the set \( K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : A(f) = x_1^2 + x_2^2 + \cdots + x_n^2 \} \). Since \( K \) is compact, there exists \( x \in \mathbb{R}^n \) adherent to \( K \).

For every \( g \in C^k(\mathbb{R}^n) \) we see that \( x \in \{ y \in \mathbb{R}^n : A(g) = g(y) \} = \{ y \in \mathbb{R}^n : A(g) = g(y) \} \). Hence \( A(g) = g(x) \).

2. Main theorems. Now we construct some differentiable real functions on a Banach space, which we will use later.

**Lemma 5.** Let \( E \) be a real Banach space, \( (x_n^*) \) a sequence in the dual space \( E^* \) of \( E \) which tends to zero in the weak* topology, \( \delta > 0 \) and \( \varphi \in C^\infty(\mathbb{R}) \) such that \( \varphi(x) = 0 \) if \( |x| < \delta \). Then we can define \( \phi(x) = \sum_{n=1}^{\infty} \varphi(x_n^*(x)) \) and we have \( \phi \in C^\infty(E) \).

**Proof.** By hypothesis \( x_n^* \) tends to zero in the weak* topology. It follows that \( \lim x_n^*(x) = 0 \) for every \( x \in E \). Moreover \( \varphi(t) = 0 \) if \( |t| < \delta \), so that \( \sum_{n=1}^{\infty} \varphi(x_n^*(x)) \) is finite and we can define \( \phi(x) \).

Now we will prove that, to each \( a \in E \), there correspond a neighborhood \( V \) of \( a \) and a natural number \( N \) such that, for every \( x \in V \), \( \phi(x) = \sum_{n=1}^{N} \varphi(x_n^*(x)) \). That is, in the neighborhood \( V \) of \( a \), we have \( \phi = \psi \circ J \) where \( J : E \to \mathbb{R}^N \) is the continuous linear map defined by \( J(x) = (x_1^*(x), \ldots, x_N^*(x)) \) and \( \psi : \mathbb{R}^N \to \mathbb{R} \) is defined in the form \( \psi(t_1, \ldots, t_N) = \sum_{i=1}^{N} \varphi(t_i) \). In other words, we will prove that \( \phi \) is a composition of infinitely differentiable functions. Therefore \( \phi \) is infinitely differentiable too.

Suppose \( a \in E \). We know that \( \lim x_n^*(a) = 0 \). Then there exists \( N \) such that \( n > N \) implies \( |x_n^*(a)| < \delta/2 \). Moreover, as \( x_n^* \to 0 \) in the weak* topology, there exists \( M \) such that \( \|x_n^*\| \leq M \) for every \( n \). Let \( V \) be the ball of center \( a \) and radius \( \delta/2M \). If \( x \in V \), \( \|x - a\| < \delta/2 \). So for every \( n > N \) we have \( |x_n^*(x)| \leq |x_n^*(x - a)| + |x_n^*(a)| < \delta/2 + \delta/2 = \delta \), and \( \varphi(x_n^*(x)) = 0 \). Thus \( \phi(x) = \sum_{n=1}^{N} \varphi(x_n^*(x)) \).

**Proposition 6.** Let \( E \) be a real and separable Banach space and \( A : C^k(E) \to \mathbb{R} \) a nonzero homomorphism. There exists \( a \in E \) such that, for every differentiable
function $\varphi$ of the form $\psi \circ J$, where $\psi \in C^\infty(\mathbb{R}^n)$ and $J : E \to \mathbb{R}^n$ is a continuous linear map, we have $A(\varphi) = \varphi(a)$.

**Proof.** Let $u : E^* \to \mathbb{R}$ be the linear form defined by $u(x^*) = A(x^*)$. First we will show that there exists $a \in E$ such that, for every $x^*$, $u(x^*) = x^*(a)$. It is the same to prove that $u$ is weak* continuous. Since $E$ is complete and separable, we apply a consequence of Grothendieck’s completeness theorem (cf. Horvath [2, p. 253]) and it remains to prove that, if $x^*_n \to 0$ in the weak* topology, then $u(x^*_n) \to 0$.

First notice that, if $x^*_1, \ldots, x^*_n \in E^*$ and $\psi \in C^\infty(\mathbb{R}^n)$, then $A(\psi(x^*_1, \ldots, x^*_n)) = \psi(u(x^*_1), \ldots, u(x^*_n))$. Indeed, let $J : E \to \mathbb{R}^n$ be the continuous linear map defined by $J(x) = (x^*_1(x), \ldots, x^*_n(x))$. There exists an homomorphism of algebras $B : C^\infty(\mathbb{R}^n) \to C^\infty(E)$ that applies $\psi$ into $\psi \circ J$. The composition $A \circ B : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a nonzero homomorphism. Thus by Proposition 4, there exists $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that $A \circ B(\psi) = \psi(\alpha_1, \ldots, \alpha_n)$, that is $A(\psi \circ J) = \psi(\alpha_1, \ldots, \alpha_n)$ for every $\psi \in C^\infty(\mathbb{R}^n)$. Now if we consider the function $\psi_i$ defined by $\psi_i(t_1, \ldots, t_n) = t_i$, we obtain $A(x^*_i) = \alpha_i = u(x^*_i)$. Therefore

$$A(\psi(x^*_1, \ldots, x^*_n)) =\, A(\psi \circ J) = A(\psi(u(x^*_1), \ldots, x^*_n))).$$

Now let $(x^*_n)$ be a sequence in $E^*$ which tends to zero in the weak* topology. We have to prove that $\lim u(x^*_n) = 0$.

Fix $\delta > 0$. Choose $\varphi \in C^\infty(\mathbb{R})$, $\varphi \geq 0$, $\varphi(t) = 0$ if $|t| < \delta$ and $\varphi(t) = 1$ if $|t| > 2\delta$. By Lemma 5, we can consider the function $\phi(x) = \sum_{n=1}^\infty \varphi(x^*_n(x)) \in C^\infty(E)$. Now for every natural number $N$, we define the function $g_N$ in the form $g_N(x) = \phi(x) - \sum_{n=1}^N \varphi(x^*_n(x))$. Since $g_N(x) \geq 0$ Proposition 1 can be applied. It shows that there exists $x$ such that $A(g_N) = g_N(x) \geq 0$. Now $(1)$ leads to

$$A(\varphi) \geq A \left( \sum_{n=1}^N \varphi(x^*_n(x)) \right) = \sum_{n=1}^N \varphi(u(x^*_n))).$$

This result being valid for every $N$ and observing that $\varphi(u(x^*_n))) \geq 0$, we conclude that the series $\sum_{n=1}^\infty \varphi(u(x^*_n)))$ converges absolutely. Hence we have $|u(x^*_n)| < 2\delta$ except for a finite number of indices. Thus the linear form $u$ is weak* continuous, so that there exists $a \in E$ such that $u(x^*) = x^*(a)$ for every $x^* \in E^*$.

Finally if $\varphi = \psi \circ J$ we obtain, by $(1)$,

$$A(\varphi) = \psi(u(x^*_1), \ldots, u(x^*_n))) = \psi(x^*_1(a), \ldots, x^*_n(a)) = \varphi(a)$$

and the proof is complete.

**Proposition 7.** Let $E$ be a real Banach space, $(x^*_n)$ a sequence in the dual space $E^*$ and $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ if $|t| > 1$. Then there exists a sequence $(\alpha_n)$ of positive real numbers so that $\phi(x) = \sum_{n=1}^\infty \alpha_n \varphi(x^*_n(x))$ defines a function $\phi \in C^\infty(E)$.

**Proof.** Fix $x^* \in E^*$ and denote by $(x^*)_n$ the multilinear map defined by $(x^*)_n(x_1, \ldots, x_n) = x^*(x_1) x^*(x_2) \cdots x^*(x_n)$ in $\mathcal{L}(E, E, \ldots, E; \mathbb{R})$. We see that the $n$th derivative of $\psi \circ x^*$ is given by $D^n(\varphi \circ x^*)(x) = D^n\varphi(x^*(x))(x^*)_n$.

Since $\varphi(t) = 1$ if $|t| > 1$ we obtain that there exist constants $M_n$ such that $|D^n\varphi(x^*(x))| \leq M_n$. Hence

$$\|D^n(\varphi \circ x^*)(x)\|_{\mathcal{L}(E, \ldots, E; \mathbb{R})} \leq M_n\|x^*\|^n.$$
In order to prove that the series $\sum \alpha_n \varphi(x^n_*(x))$ converges, it suffices to choose the sequence $(\alpha_n)$ such that $\sum \left| \alpha_n \right| M_0 < +\infty$. Moreover the function $\phi$ is differentiable if we choose the sequence $(\alpha_n)$ such that $\sum \left| \alpha_n \right| M_1 \|x^n_*(x)\|$ converges, and $\phi$ is of class $C^\infty$ if for every natural number $p$ the series $\sum \left| \alpha_n \right| M_p \|x^n_*(x)\|^p$ converges. But it is obvious that we can choose the sequence $(\alpha_n)$ verifying these conditions, which completes the proof.

**THEOREM 8.** Let $E$ be a real and separable Banach space and $A : C^k(E) \rightarrow \mathbb{R}$ a nonzero homomorphism. Then there exists a point $a \in E$ such that $A(\varphi) = \varphi(a)$ for every $\varphi \in C^k(E)$.

**PROOF.** We know, by Proposition 6, that there exists $a \in E$ such that, if $\varphi \in C^\infty(E)$ is of the form $\varphi = \psi \circ J$, where $\psi \in C^\infty(\mathbb{R}^n)$ and $J : E \rightarrow \mathbb{R}^n$ is a continuous linear map, then $A(\varphi) = \varphi(a)$.

Since $E$ is separable, there exists a sequence $(x^n_*)$ in the dual space such that $\|x^n_*\| = \sup \left| x^n_*(x) \right|$ for every $x \in E$. To prove this, let $(a_n)$ be a dense sequence in $E$ and choose $x^n_*$ in the dual space such that $\|x^n_*\| = 1$ and $x^n_*(a_n) = \|a_n\|$. Now $\|x^n_*(x)\| \leq \|x\|$ for every $n$ and, on the other hand, to each $\varepsilon > 0$, there corresponds $a_n$ such that $\|x - a_n\| < \varepsilon$. Thus we have $\|x\| \leq \|a_n\| + \|x - a_n\| \leq \varepsilon + x^n_*(a_n) \leq 2\varepsilon + x^n_*(x)$.

Now let $f \in C^k(E)$. As $f$ is continuous in $a$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - a\| \leq \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Next we choose a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(t) \neq 0$ if $|t| > \delta$, $\varphi(t) = 0$ if $|t| \leq \delta$, $\varphi(t) = 1$ if $|t| > 1$, and $\varphi(t) \geq 0$ for every $t \in \mathbb{R}$. Now Proposition 7 can be applied. It shows that there exists a sequence $(\alpha_n)$ of positive real numbers such that the function $\psi(x) = \sum_{n=1}^{\infty} \alpha_n \varphi(x^n_*(x - a)) \in C^\infty(E)$. In the same way the function $\phi(x) = \sum_{n=1}^{\infty} (\alpha_n/n) \varphi(x^n_*(x - a)) \in C^\infty(E)$.

Observe that, for $\|x - a\| \leq \delta$, $\phi(x) = 0$ since $|x^n_*(x - a)| < \delta$. Moreover, if $\|x - a\| > \delta$, there exists a natural number $n$ such that $|x^n_*(x - a)| > \delta$, so that $\phi(x) > 0$.

Our next objective is to prove that $A(\phi) = 0$. Fix $N$. By Proposition 2, there exists $x \in E$ such that $A(\phi) = \phi(x)$, $A(x^n_*(a)) = x^n_*(x)$ for every $i = 1, 2, \ldots, N$ and $A(\psi) = \psi(x)$. It follows that

$$A(\psi) = \psi(x) = \sum_{n=1}^{\infty} \alpha_n \varphi(x^n_*(x - a)) = \sum_{n=1}^{\infty} \alpha_n \varphi(x^n_*(x - a)) \geq 0$$

and

$$A(\phi) = \phi(x) = \sum_{n=1}^{\infty} (\alpha_n/n) \varphi(x^n_*(x - a)) = \sum_{n=1}^{\infty} (\alpha_n/n) \varphi(x^n_*(x - a))$$

Therefore $0 \leq NA(\phi) \leq A(\psi)$. As this is valid for every $N$, we conclude that $A(\phi) = 0$.

Finally, by Proposition 2, there exists $y \in E$ such that $A(f) = f(y)$ and $0 = A(\phi) = \phi(y)$. Since $\phi(y) = 0$, we have $\|y - a\| < \delta$. It turns out, by the choice of $\delta$, that $|f(y) - f(a)| < \varepsilon$. Hence what we have proved is that, for every $\varepsilon > 0$, the inequality $|A(f) - f(a)| < \varepsilon$ holds. This completes the proof.
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