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A REAL VALUED HOMOMORPHISM ON ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. In this paper we prove that, for every homomorphism A on $C^k(E)$, there exists $x \in E$ such that $A(f) = f(x)$ for $f \in C^k(E)$. Here $C^k(E)$ $(k = 1, 2, \ldots, \infty)$ denotes the algebra of all k-times differentiable real functions on a real and separable Banach space E.

Introduction. Let $C^k(E)$ $(k = 1, 2, \ldots, \infty)$ be the algebra of all k-times differentiable function $f\colon E \to \mathbf{R}$ defined on a real Banach space E. Since $C^k(E)$ is an algebra of functions it makes sense to ask if, to each nonzero homomorphism $A\colon C^k(E) \to \mathbf{R}$, there corresponds $x \in E$ such that $A(f) = f(x)$ for all $f \in C^k(E)$, in other words, if A is an evaluation.

If $k = 0$, the answer is affirmative if and only if E is realcompact. It turns out, since E is a metric space, that it is realcompact if and only if card(E) is nonmeasurable (cf. Gillman and Jerison [**1**, pp. 226–232]).

On the other hand Jaramillo [**4**] defining two topologies τ_ω^k and τ_δ^k on $C^k(E)$, have proved that, if $\tau_\omega^k = \tau_\delta^k$, then every homomorphism on $C^k(E)$ is an evaluation. They have also proved that $\tau_\omega^k = \tau_\delta^k$ if E has nonmeasurable cardinal and E admits B^k partitions of unity. To see what kind of Banach spaces admits B^k partitions of unity, see Wells [**6**] and Sundaresan and Swaminatan [**5**].

In this paper we generalize the result of Jaramillo to all separable Banach spaces. We give a direct proof without using any topology on $C^k(E)$ nor partitions of unity on E.

1. Associated filter to a real homomorphism. In the following, $A\colon C^k(E) \to \mathbf{R}$ denotes a nonzero homomorphism. For $\alpha \in \mathbf{R}$, let $\hat{\alpha}$ be the constant function that associates the value α to each element $x \in E$. We claim that $A(\hat{\alpha}) = \alpha$ for every $\alpha \in \mathbf{R}$. Indeed, $A(\hat{1}) \neq 0$ because $A \neq 0$. Then $B\colon \mathbf{R} \to \mathbf{R}$ defined by $B(\alpha) = A(\hat{\alpha})$ is a nonzero homomorphism from $\mathbf{R}$ into $\mathbf{R}$. Therefore B is the identity homomorphism, that is $A(\hat{\alpha}) = \alpha$ for every $\alpha \in \mathbf{R}$.

PROPOSITION 1. *Let $A\colon C^k(E) \to \mathbf{R}$ be a nonzero homomorphism. For every $f \in C^k(E)$, the set $\{x \in E\colon A(f) = f(x)\}$ is nonempty.*

PROOF. Suppose $\{x \in E\colon A(f) = f(x)\} = \varnothing$. Then $f - \widehat{A(f)}$ is a real function, which does not take the value 0. So $(f - \widehat{A(f)})^{-1} \in C^k(E)$ and we see that

$$1 = A(\hat{1}) = A((f - \widehat{A(f)})(f - \widehat{A(f)})^{-1}) = A(f - \widehat{A(f)})A((f - \widehat{A(f)})^{-1})$$

which contradicts $A(f - \widehat{A(f)}) = A(f) - A(f) = 0$.

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PROPOSITION 2. *Let $A\colon C^k(E) \to \mathbf{R}$ be a nonzero homomorphism. Denote by $S(f_1, \dots, f_n)$ the set $\{x \in E\colon A(f_i) = f_i(x),\ i = 1, 2, \dots, n\}$. The collection $\mathfrak{S}$ of all the sets $S(f_1, \dots, f_n)$ where $f_1, \dots, f_n \in C^k(E)$ is a filter basis on E.*

PROOF. Since $S(f_1, \dots, f_n) \cap S(g_1, \dots, g_m) = S(f_1, \dots, f_n, g_1, \dots, g_m)$, it suffices to prove that $S(f_1, \dots, f_n) \neq \varnothing$ for every $f_1, \dots, f_n \in C^k(E)$.

Consider the function $g = \sum_{i=1}^n (f_i - \widehat{A(f_i)})^2 \in C^k(E)$. By Proposition 1, there exists $x \in E$ such that $A(g) = g(x)$. From the definition of g we get $A(g) = 0$, so that $\sum (f_i(x) - A(f_i))^2 = 0$ which forces $f_i(x) = A(f_i)$ for every $i = 1, 2, \dots, n$.

DEFINITION 3. *Let $A\colon C^k(E) \to \mathbf{R}$ be a nonzero homomorphism. We will say that the filter $\mathscr{F}$ generated by the filter basis $\mathfrak{S}$, from Proposition 2, is the filter associated to A.*

PROPOSITION 4. *Let $A\colon C^k(\mathbf{R}^n) \to \mathbf{R}$ be a nonzero homomorphism. Then there exists $x \in \mathbf{R}^n$ such that for every $f \in C^k(\mathbf{R}^n), A(f) = f(x)$.*

PROOF. Let $f \in C^k(\mathbf{R}^n)$ be the function defined by $f(x_1, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$. The filter $\mathscr{F}$ associated to A contains the set $K = \{(x_1, \dots, x_n) \in \mathbf{R}^n\colon A(f) = x_1^2 + x_2^2 + \dots + x_n^2\}$. Since K is compact, there exists $x \in \mathbf{R}^n$ adherent to $\mathscr{F}$.

For every $g \in C^k(\mathbf{R}^n)$ we see that $x \in \overline{\{y \in R^n\colon A(g) = g(y)\}} = \{y \in \mathbf{R}^n\colon A(g) = g(y)\}$. Hence $A(g) = g(x)$.

2. Main theorems. Now we construct some differentiable real functions on a Banach space, which we will use later.

LEMMA 5. *Let E be a real Banach space, (x_n^*) a sequence in the dual space E^* of E which tends to zero in the weak* topology, $\delta > 0$ and $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(x) = 0$ if $|x| < \delta$. Then we can define $\phi(x) = \sum_{n=1}^\infty \varphi(x_n^*(x))$ and we have $\phi \in C^\infty(E)$.*

PROOF. By hypothesis x_n^* tends to zero in the weak* topology. It follows that $\lim x_n^*(x) = 0$ for every $x \in E$. Moreover $\varphi(t) = 0$ if $|t| < \delta$, so that $\sum_{n=1}^\infty \varphi(x_n^*(x))$ is finite and we can define $\phi(x)$.

Now we will prove that, to each $a \in E$, there correspond a neighborhood V of a and a natural number N such that, for every $x \in V$, $\phi(x) = \sum_{n=1}^N \varphi(x_n^*(x))$. That is, in the neighborhood V of a, we have $\phi = \psi \circ J$ where $J\colon E \to \mathbf{R}^N$ is the continuous linear map defined by $J(x) = (x_1^*(x), \dots, x_N^*(x))$ and $\psi\colon \mathbf{R}^N \to \mathbf{R}$ is defined in the form $\psi(t_1, \dots, t_N) = \sum_{i=1}^N \varphi(t_i)$. In other words, we will prove that ϕ is a composition of infinitely differentiable functions. Therefore ϕ is infinitely differentiable too.

Suppose $a \in E$. We know that $\lim x_n^*(a) = 0$. Then there exists N such that $n > N$ implies $|x_n^*(a)| < \delta/2$. Moreover, as $x_n^* \to 0$ in the weak* topology, there exists M such that $\|x_n^*\| \leq M$ for every n. Let V be the ball of center a and radius $\delta/2M$. If $x \in V$, $\|x - a\| < \delta/2M$. So for every $n > N$ we have $|x_n^*(x)| \leq |x_n^*(x - a)| + |x_n^*(a)| < \delta/2 + \delta/2 = \delta$, and $\varphi(x_n^*(x)) = 0$. Thus $\phi(x) = \sum_{n=1}^N \varphi(x_n^*(x))$.

PROPOSITION 6. *Let E be a real and separable Banach space and $A\colon C^k(E) \to \mathbf{R}$ a nonzero homomorphism. There exists $a \in E$ such that, for every differentiable*

function φ of the form $\psi \circ J$, where $\psi \in C^\infty(\mathbb{R}^n)$ and $J\colon E \to \mathbb{R}^n$ is a continuous linear map, we have $A(\varphi) = \varphi(a)$.

PROOF. Let $u\colon E^* \to \mathbb{R}$ be the linear form defined by $u(x^*) = A(x^*)$. First we will show that there exists $a \in E$ such that, for every x^*, $u(x^*) = x^*(a)$. It is the same to prove that u is weak* continuous. Since E is complete and separable, we apply a consequence of Grothendieck's completeness theorem (cf. Horvath [**2**, p. 253]) and it remains to prove that, if $x_n^* \to 0$ in the weak* topology, then $u(x_n^*) \to 0$.

First notice that, if $x_1^*, \dots, x_n^* \in E^*$ and $\psi \in C^\infty(\mathbb{R}^n)$, then $A(\psi(x_1^*(x), \dots, x_n^*(x)) = \psi(u(x_1^*), \dots, u(x_n^*))$. Indeed, let $J\colon E \to \mathbb{R}^n$ be the continuous linear map defined by $J(x) = (x_1^*(x), \dots, x_n^*(x))$. There exists an homomorphism of algebras $B\colon C^\infty(\mathbb{R}^n) \to C^\infty(E)$ that applies ψ into $\psi \circ J$. The composition $A \circ B\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a nonzero homomorphism. Thus by Proposition 4, there exists $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that $A \circ B(\psi) = \psi(\alpha_1, \dots, \alpha_n)$, that is $A(\psi \circ J) = \psi(\alpha_1, \dots, \alpha_n)$ for every $\psi \in C^\infty(\mathbb{R}^n)$. Now if we consider the function ψ_i defined by $\psi_i(t_1, \dots, t_n) = t_i$ we obtain $A(x_i^*) = \alpha_i = u(x_i^*)$. Therefore

$$A(\psi(x_1^*(x), \dots, x_n^*(x)) = A(\psi \circ J) = \psi(u(x_1^*), \dots, u(x_n^*)). \tag{1}$$

Now let (x_n^*) be a sequence in E^* which tends to zero in the weak* topology. We have to prove that $\lim u(x_n^*) = 0$.

Fix $\delta > 0$. Choose $\varphi \in C^\infty(\mathbb{R})$, $\varphi \geq 0$, $\varphi(t) = 0$ if $|t| < \delta$ and $\varphi(t) = 1$ if $|t| > 2\delta$. By Lemma 5, we can consider the function $\phi(x) = \sum_{n=1}^\infty \varphi(x_n^*(x)) \in C^\infty(E)$. Now for every natural number N, we define the function g_N in the form $g_N(x) = \phi(x) - \sum_{n=1}^N \varphi(x_n^*(x))$. Since $g_N(x) \geq 0$ Proposition 1 can be applied. It shows that there exists $\bar{x}$ such that $A(g_N) = g_N(\bar{x}) \geq 0$. Now (1) leads to

$$A(\phi) \geq A\left(\sum_{n=1}^N \varphi(x_n^*(x))\right) = \sum_{n=1}^N \varphi(u(x_n^*)).$$

This result being valid for every N and observing that $\varphi(u(x_n^*)) \geq 0$, we conclude that the series $\sum_{n=1}^\infty \varphi(u(x_n^*))$ converges absolutely. Hence we have $|u(x_n^*)| < 2\delta$ except for a finite number of indices. Thus the linear form u is weak* continuous, so that there exists $a \in E$ such that $u(x^*) = x^*(a)$ for every $x^* \in E^*$.

Finally if $\varphi = \psi \circ J$ we obtain, by (1),

$$A(\varphi) = \psi(u(x_1^*), \dots, u(x_n^*)) = \psi(x_1^*(a), \dots, x_n^*(a)) = \varphi(a)$$

and the proof is complete.

PROPOSITION 7. *Let E be a real Banach space, (x_n^*) a sequence in the dual space E^* and $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ if $|t| > 1$. Then there exists a sequence (α_n) of positive real numbers so that $\phi(x) = \sum_{n=1}^\infty \alpha_n \varphi(x_n^*(x))$ defines a function $\phi \in C^\infty(E)$.*

PROOF. Fix $x^* \in E^*$ and denote by $(x^*)^n$ the multilinear map defined by $(x^*)^n(x_1, \dots, x_n) = x^*(x_1)x^*(x_2)\cdots x^*(x_n)$ in $\mathscr{L}(E, E, \dots, E; \mathbb{R})$. We see that the nth derivative of $\varphi \circ x^*$ is given by $D^n(\varphi \circ x^*)(x) = D^n\varphi(x^*(x))(x^*)^n$.

Since $\varphi(t) = 1$ if $|t| > 1$ we obtain that there exist constants M_n such that $|D^n\varphi(x^*(x))| \leq M_n$. Hence

$$\|D^n(\varphi \circ x^*)(x)\|_{\mathscr{L}(E,\dots,E;\mathbb{R})} \leq M_n \|x^*\|^n.$$

In order to prove that the series $\sum \alpha_n \varphi(x_n^*(x))$ converges, it suffices to choose the sequence (α_n) such that $\sum |\alpha_n| M_0 < +\infty$. Moreover the function ϕ is differentiable if we choose the sequence (α_n) such that series $\sum |\alpha_n| M_1 \|x_n^*\|$ converges, and ϕ is of class C^∞ if for every natural number p the series $\sum |\alpha_n| M_p \|x_n^*\|^p$ converges. But it is obvious that we can choose the sequence (α_n) verifying these conditions, which completes the proof.

THEOREM 8. *Let E be a real and separable Banach space and $A\colon C^k(E) \to \mathbb{R}$ a nonzero homomorphism. Then there exists a point $a \in E$ such that $A(\varphi) = \varphi(a)$ for every $\varphi \in C^k(E)$.*

PROOF. We know, by Proposition 6, that there exists $a \in E$ such that, if $\varphi \in C^\infty(E)$ is of the form $\varphi = \psi \circ J$, where $\psi \in C^\infty(\mathbb{R}^n)$ and $J\colon E \to \mathbb{R}^n$ is a continuous linear map, then $A(\varphi) = \varphi(a)$.

Since E is separable, there exists a sequence (x_n^*) in the dual space such that $\|x\| = \sup |x_n^*(x)|$ for every $x \in E$. To prove this, let (a_n) be a dense sequence in E and choose x_n^* in the dual space such that $\|x_n^*\| = 1$ and $x_n^*(a_n) = \|a_n\|$. Now $|x_n^*(x)| \le \|x\|$ for every n and, on the other hand, to each $\varepsilon > 0$, there corresponds a_n such that $\|x - a_n\| < \varepsilon$. Thus we have $\|x\| \le \|a_n\| + \|x - a_n\| \le \varepsilon + x_n^*(a_n) \le 2\varepsilon + |x_n^*(x)|$.

Now let $f \in C^k(E)$. As f is continuous in a, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - a\| \le \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Next we choose a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(t) \ne 0$ if $|t| > \delta$, $\varphi(t) = 0$ if $|t| \le \delta$, $\varphi(t) = 1$ if $|t| > 1$, and $\varphi(t) \ge 0$ for every $t \in \mathbb{R}$. Now Proposition 7 can be applied. It shows that there exists a sequence (α_n) of positive real numbers such that the function $\psi(x) = \sum_{n=1}^\infty \alpha_n \varphi(x_n^*(x - a)) \in C^\infty(E)$. In the same way the function $\phi(x) = \sum_{n=1}^\infty (\alpha_n/n) \varphi(x_n^*(x - a)) \in C^\infty(E)$.

Observe that, for $\|x - a\| \le \delta$, $\phi(x) = 0$ since $|x_n^*(x - a)| < \delta$. Moreover, if $\|x - a\| > \delta$, there exists a natural number n such that $|x_n^*(x - a)| > \delta$, so that $\phi(x) > 0$.

Our next objective is to prove that $A(\phi) = 0$. Fix N. By Proposition 2, there exists $x \in E$ such that $A(\phi) = \phi(x)$, $A(x_i^*) = x_i^*(a) = x_i^*(x)$ for every $i = 1, 2, \ldots, N$ and $A(\psi) = \psi(x)$. It follows that

$$A(\psi) = \psi(x) = \sum_{n=1}^{\infty} \alpha_n \varphi(x_n^*(x - a)) = \sum_{N+1}^{\infty} \alpha_n \varphi(x_n^*(x - a)) \ge 0$$

and

$$A(\phi) = \phi(x) = \sum_{n=1}^{\infty} (\alpha_n/n) \varphi(x_n^*(x - a)) = \sum_{N+1}^{\infty} (\alpha_n/n) \varphi(x_n^*(x - a)).$$

Therefore $0 \le NA(\phi) \le A(\psi)$. As this is valid for every N, we conclude that $A(\phi) = 0$.

Finally, by Proposition 2, there exists $y \in E$ such that $A(f) = f(y)$ and $0 = A(\phi) = \phi(y)$. Since $\phi(y) = 0$, we have $\|y - a\| < \delta$. It turns out, by the choice of δ, that $|f(y) - f(a)| < \varepsilon$. Hence what we have proved is that, for every $\varepsilon > 0$, the inequality $|A(f) - f(a)| < \varepsilon$ holds. This completes the proof.

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