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ON A THEOREM OF HARDY AND LITTLEWOOD

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ABSTRACT. In this paper, we give an extension of a classical theorem of Hardy and Littlewood on power series. Let φ be a strictly positive function defined on some interval $(\delta, 1)$, satisfying a certain condition of limit. We prove that if $f(x)$ is the sum of a convergent power series for $0 < x < 1$ with nonnegative coefficients a_n and $f(x) \sim \varphi(x)$ $(x \to 1)$, then $S_n \sim \alpha \cdot \varphi(x_0^{1/n})$ $(n \to \infty)$, where $S_n = a_0 + a_1 + \cdots + a_n$, $x_0 \in (0,1)$ and α depends only upon φ.

It is classical the following result about power series: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a series converging for $0 < x < 1$, with $a_n \geq 0$ $(n = 0, 1, 2, \dots)$. Assume that $\lim_{x \to 1}(1-x) \cdot f(x) = 1$. Then $\lim_{n\to\infty}(S_n/n) = 1$, where $S_n = \sum_{j=0}^{n} a_j$. This theorem is due to Hardy and Littlewood (see [**2**] and [**1**, pp. 481–486]). In 1930, Karamata [**3**] gave an elegant proof of the result, based upon the Weierstrass theorem of uniform approximation. See also [**4**, pp. 226–229]. In this note we use the same idea to derive an extension of the Hardy-Littlewood theorem. We substitute the function $(1-x)^{-1}$ such that $f(x) \sim (1-x)^{-1}$ $(x \to 1)$ by a more general function $\varphi(x)$. The conclusion is similar. We shall obtain that $\lim_{n\to\infty}(S_n/\varphi(x_0^{1/n})) = \alpha$, where $0 < x_0 < 1$ and $0 \leq \alpha < +\infty$, α depending only upon φ.

First, we state a result which we shall need in proving our theorem. Denote by R the real line. A real sequence $\{c_m : m = 0, 1, 2, \dots\}$ is said to be completely monotonic if

$$c_m \geq 0 \text{ and } (-1)^n \cdot \Delta^n c_m \geq 0 \qquad (m, n = 0, 1, 2, \dots). \tag{1}$$

Here, Δ^n $(n = 0, 1, 2, \dots)$ denote the successive differences.

LEMMA. *Let $\{c_m : m = 0, 1, 2, \dots\}$ be a real sequence. Then there is a nondecreasing function (resp. a function of bounded variation) $\sigma : [0,1] \to R$ such that $\{c_m : m = 0, 1, 2, \dots\}$ is the moment sequence for σ, i.e., $c_m = \int_0^1 x^m \, d\sigma(x)$ $(m = 0, 1, 2, \dots)$, if and only if $\{c_m : m = 0, 1, 2, \dots\}$ is completely monotonic (resp. $\{c_m : m = 0, 1, 2, \dots\}$ is the difference of two completely monotonic sequences).*

For the proof, see [**5**, pp. 100–109]. In both cases, the function σ is unique, if we identify two functions σ', σ'' for which $\int_0^1 f \, d\sigma' = \int_0^1 f \, d\sigma''$ for every continuous function f on $[0,1]$. We shall say that σ generates $\{c_m\}$. It is known that the set $C(h) = \{x \in (0,1) : h \text{ is continuous at } x\}$ is conumerable in $(0,1)$, if h is of bounded variation on $[0,1]$. Next, we state our result.

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THEOREM. *Let $\varphi(x)$ be a strictly positive function defined on some interval $(\delta, 1)$ $(\delta < 1)$, and let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a series converging for $0 < x < 1$, with $a_n \geq 0$ $(n = 0, 1, 2, \ldots)$. Assume that $f(x) \sim \varphi(x)$ $(x \to 1)$ and that the limits $c_m = \lim_{x \to 1} \varphi(x^{m+1})/\varphi(x)$ $(m = 0, 1, 2, \ldots)$ exist as real numbers. In addition, assume that $\{c_m : m = 0, 1, 2, \ldots\}$ is the difference of two completely monotonic sequences. If σ generates $\{c_m\}$ and $x_0 \in C(\sigma)$, then*

$$\lim_{n \to \infty} \frac{S_n}{\varphi(x_0^{1/n})} = \alpha,$$

where $\alpha = \int_{x_0}^1 t^{-1}\, d\sigma(t)$.

PROOF. Consider the function $g(t)$ defined on $[0, 1]$ by $g(t) = 0$ $(0 \leq t < x_0)$, $g(t) = 1/t$ $(x_0 \leq t \leq 1)$. Let ε be a given positive number. Denote by $V(t)$ the total variation of σ on $[0, t]$ $(0 \leq t \leq 1)$, and choose β such that

$$0 < \beta < \min(x_0, 1 - x_0)$$

and

$$\max(V(x_0) - V(x_0 - \beta), V(x_0 + \beta) - V(x_0)) < (\varepsilon/2)x_0.$$

This is possible because σ is continuous at x_0. Define the functions $r(t)$ and $s(t)$ by $r(t) = 0$ $(0 \leq t < x_0 - \beta)$, $r(t) = (x_0 \cdot \beta)^{-1} \cdot t - 1/\beta + 1/x_0$ $(x_0 - \beta \leq t < x_0)$, $r(t) = 1/t$ $(x_0 \leq t \leq 1)$, $s(t) = 0$ $(0 \leq t < x_0)$, $s(t) = (x_0 \cdot \beta + \beta^2)^{-1} \cdot t - x_0 \cdot (x_0 \cdot \beta + \beta^2)^{-1}$ $(x_0 \leq t < x_0 + \beta)$, $s(t) = 1/t$ $(x_0 + \beta \leq t \leq 1)$. Then r and s are continuous on $[0, 1]$. From the Weierstrass theorem, we may choose polynomials $p(x)$, $P(x)$ such that $0 \leq s(x) - p(x) < \varepsilon \cdot (1 + 2V(1))^{-1}$ and $0 \leq P(x) - r(x) < \varepsilon \cdot (1 + 2V(1))^{-1}$ $(0 \leq x \leq 1)$. Then, evidently, $p(x) \leq s(x) \leq g(x)$ and $g(x) \leq r(x) \leq P(x)$ $(0 \leq x \leq 1)$. Also, $\int_0^1 (r(t) - g(t))\, d\sigma(t) = \int_{x_0 - \beta}^{x_0} r(t)\, d\sigma(t) \leq (V(x_0) - V(x_0 - \beta))/x_0 < \varepsilon/2$. But $\int_0^1 P(t)\, d\sigma(t) < \varepsilon/2 + \int_0^1 r(t)\, d\sigma(t)$, so

$$\int_0^1 P(t)\, d\sigma(t) < \varepsilon + \int_0^1 g(t)\, d\sigma(t). \tag{2}$$

Analogously, $\int_0^1 (g(t) - s(t))\, d\sigma(t) \leq \int_{x_0}^{x_0 + \beta} (g(t) - s(t))\, d\sigma(t) \leq (V(x_0 + \beta) - V(x_0))/x_0 < \varepsilon/2$. From $\int_0^1 s(t)\, d\sigma(t) < \varepsilon/2 + \int_0^1 p(t)\, d\sigma(t)$, we obtain

$$\int_0^1 g(t)\, d\sigma(t) < \varepsilon + \int_0^1 p(t)\, d\sigma(t). \tag{3}$$

Fix $m \in \{0, 1, 2, \ldots\}$. Then $(1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^{n+mn} = (\varphi(x^{m+1})/\varphi(x)) \cdot ((\sum_{n=0}^{\infty} a_n (x^{m+1})^n)/\varphi(x^{m+1}))$ which tends to c_m when $x \to 1$, by hypothesis. But $c_m = \int_0^1 x^m\, d\sigma(x)$. Then, by linearity, $\lim_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot Q(x^n) = \int_0^1 Q(x)\, d\sigma(x)$ for all polynomials $Q(x)$. Now, since $g(x) \leq P(x)$ and $\varphi(x)$ and the a_n $(n = 0, 1, 2, \ldots)$ are positive, we have $\limsup_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot g(x^n) \leq \limsup_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot P(x^n) = \int_0^1 P(t)\, d\sigma(t) < \int_0^1 g(t)\, d\sigma(t) + \varepsilon$, accordingly with (2). By doing $\varepsilon \to 0$, we obtain $\limsup_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot g(x^n) \leq \int_0^1 g(t)\, d\sigma(t)$. If we use $p(x)$ instead of $P(x)$, it follows similarly from (3) that $\liminf_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot g(x^n) \geq \int_0^1 g(t)\, d\sigma(t)$. Consequently,

$\lim_{x\to 1}(1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \cdot g(x^n) = \int_0^1 g(t)\,d\sigma(t) = \int_{x_0}^1 t^{-1}\,d\sigma(t) = \alpha$. Let $x_j = x_0^{1/j}$ $(j = 1, 2, \dots)$. Then $\sum_{n=0}^{\infty} a_n x_j^n \cdot g(x_j^n) = \sum_{n=0}^{j} a_n = S_j$, because

$$\begin{aligned} g(x_j^n) &= 1/x_j^n \quad \text{iff} \quad x_j^n \geq x_0 \quad \text{iff} \quad n \leq j \\ &= 0 \quad \text{iff} \quad x_j^n < x_0 \quad \text{iff} \quad n > j. \end{aligned}$$

Finally, $x_j \in (\delta, 1)$ for $j \geq j_0$ and $\lim_{j\to\infty}(1/\varphi(x_j))\cdot \sum_{n=0}^{\infty} a_n x_j^n \cdot g(x_j^n) = \alpha$. Hence $\lim_{j\to\infty} S_j/\varphi(x_0^{1/j}) = \alpha$. The theorem is proved.

It is clear that the Hardy-Littlewood theorem is a special case of this. In the classical theorem, $c_m = 1/(m+1)$ and $\sigma(x) \equiv x$. The point $x_0 \in (0,1)$ is arbitrary.

We give an easy example. Let $\varphi(x) = \log((1-x)^{-1})$. Then $\varphi(x_0^{1/j}) \sim \log j$ $(j \to \infty)$ for each $x_0 \in (0,1)$ and $c_m = \lim_{x\to 1}(\log(1 - x^{m+1}))/\log(1-x) = 1$ $(m = 0, 1, 2, \dots)$. Then the sequence $\{c_m\}$ is completely monotonic, because it satisfies (1) trivially. The function $\sigma(x) = 0$ $(0 \leq x < 1)$, $\sigma(1) = 1$ is of bounded variation, continuous at every $x_0 \in (0,1)$ and satisfies $c_m = \int_0^1 t^m\,d\sigma(t)$ $(m = 0, 1, 2, \dots)$. In addition, $\alpha = \int_{x_0}^1 1/t\,d\sigma(t) = 1$. Hence, we have: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $(0 \leq x < 1)$, $a_n \geq 0$ $(n = 0, 1, 2, \dots)$ and $\lim_{x\to 1} f(x)/(\log(1-x)^{-1})) = 1$, then $S_n \sim \log n$ $(n \to \infty)$.

REMARK. In [**5**, pp. 109–112], one can find several conditions which guarantee that we may choose σ to be continuous at each $x_0 \in (0,1)$. For the sake of completeness, we reproduce them:

(a) There exists a constant L such that

$$(1+k)\cdot|\lambda_{k,m}| < L \qquad (k, m = 0, 1, 2, \dots; m \leq k).$$

(b) There exist a number $p > 1$ and a constant L such that

$$(1+k)^{p-1}\cdot \sum_{m=0}^{k} |\lambda_{k,m}|^p < L \qquad (k = 0, 1, 2, \dots).$$

Here, $\lambda_{k,m} = \binom{k}{m}(-1)^{k-m}\cdot \Delta^{k-m} c_m$.

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