ON A THEOREM OF HARDY AND LITTLEWOOD

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ABSTRACT. In this paper, we give an extension of a classical theorem of Hardy and Littlewood on power series. Let $\varphi$ be a strictly positive function defined on some interval $(\delta, 1)$, satisfying a certain condition of limit. We prove that if $f(x)$ is the sum of a convergent power series for $0 < x < 1$ with nonnegative coefficients $a_n$ and $f(x) \sim \varphi(x)$ $(x \to 1)$, then $S_n \sim \alpha \cdot \varphi(x_0^{1/n})$ $(n \to \infty)$, where $S_n = a_0 + a_1 + \cdots + a_n$, $x_0 \in (0, 1)$ and $\alpha$ depends only upon $\varphi$.

It is classical the following result about power series: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a series converging for $0 < x < 1$, with $a_n \geq 0$ $(n = 0, 1, 2, \ldots)$. Assume that $\lim_{x \to 1} (1 - x) \cdot f(x) = 1$. Then $\lim_{n \to \infty} (S_n/n) = 1$, where $S_n = \sum_{j=0}^{n} a_j$. This theorem is due to Hardy and Littlewood (see [2] and [1, pp. 481-486]). In 1930, Karamata [3] gave an elegant proof of the result, based upon the Weierstrass theorem of uniform approximation. See also [4, pp. 226-229]. In this note we use the same idea to derive an extension of the Hardy-Littlewood theorem. We substitute the function $(1-x)^{-1}$ such that $f(x) \sim (1-x)^{-1}$ $(x \to 1)$ by a more general function $\varphi(x)$. The conclusion is similar. We shall obtain that $\lim_{n \to \infty} (S_n/\varphi(x_0^{1/n})) = \alpha$, where $0 < x_0 < 1$ and $0 \leq \alpha < +\infty$, $\alpha$ depending only upon $\varphi$.

First, we state a result which we shall need in proving our theorem. Denote by $R$ the real line. A real sequence $\{c_m: m = 0, 1, 2, \ldots\}$ is said to be completely monotonic if

$$ c_m \geq 0 \text{ and } (-1)^n \cdot \Delta^n c_m \geq 0 \quad (m, n = 0, 1, 2, \ldots). $$

Here, $\Delta^n$ $(n = 0, 1, 2, \ldots)$ denote the successive differences.

LEMMA. Let $\{c_m: m = 0, 1, 2, \ldots\}$ be a real sequence. Then there is a non-decreasing function (resp. a function of bounded variation) $\sigma: [0, 1] \to R$ such that $\{c_m: m = 0, 1, 2, \ldots\}$ is the moment sequence for $\sigma$, i.e., $c_m = \int_{0}^{1} x^m \, d\sigma(x)$ $(m = 0, 1, 2, \ldots)$, if and only if $\{c_m: m = 0, 1, 2, \ldots\}$ is completely monotonic (resp. $\{c_m: m = 0, 1, 2, \ldots\}$ is the difference of two completely monotonic sequences).

For the proof, see [5, pp. 100-109]. In both cases, the function $\sigma$ is unique, if we identify two functions $\sigma'$, $\sigma''$ for which $\int_{0}^{1} f \, d\sigma' = \int_{0}^{1} f \, d\sigma''$ for every continuous function $f$ on $[0, 1]$. We shall say that $\sigma$ generates $\{c_m\}$. It is known that the set $C(h) = \{x \in (0, 1): h$ is continuous at $x\}$ is countable in $(0, 1)$, if $h$ is of bounded variation on $[0, 1]$. Next, we state our result.

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THEOREM. Let \( \varphi(x) \) be a strictly positive function defined on some interval \((\delta, 1)\) \((\delta < 1)\), and let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be a series converging for \( 0 < x < 1 \), with \( a_n \geq 0 \) \((n = 0, 1, 2, \ldots)\). Assume that \( f(x) \sim \varphi(x) \) \((x \to 1)\) and that the limits \( c_m = \lim_{x \to 1} \varphi(x^{m+1})/\varphi(x) \) \((m = 0, 1, 2, \ldots)\) exist as real numbers. In addition, assume that \( \{c_m; m = 0, 1, 2, \ldots\} \) is the difference of two completely monotonic sequences. If \( \sigma \) generates \( \{c_m\} \) and \( x_0 \in C(\sigma) \), then
\[
\lim_{n \to \infty} \frac{S_n}{\varphi(x_0^{1/n})} = \alpha,
\]
where \( \alpha = \int_{x_0}^{1} t^{-1} \, d\sigma(t) \).

PROOF. Consider the function \( g(t) \) defined on \([0, 1]\) by \( g(t) = 0 \) \((0 < t < x_0)\), \( g(t) = 1/t \) \((x_0 \leq t \leq 1)\). Let \( \varepsilon \) be a given positive number. Denote by \( V(t) \) the total variation of \( \sigma \) on \([0, t]\) \((0 \leq t \leq 1)\), and choose \( \beta \) such that
\[
0 < \beta < \min(x_0, 1 - x_0)
\]
and
\[
\max(V(x_0) - V(x_0 - \beta), V(x_0 + \beta) - V(x_0)) < (\varepsilon/2)x_0.
\]
This is possible because \( \sigma \) is continuous at \( x_0 \). Define the functions \( r(t) \) and \( s(t) \) by
\[
r(t) = 0 \quad (0 < t < x_0 - \beta), \quad r(t) = (x_0 - \beta)^{-1} \cdot (1 - 1/\beta + 1/x_0) \quad (x_0 - \beta < t < x_0), \quad r(t) = 1/t \quad (x_0 < t < 1),
\]
and
\[
s(t) = (x_0 - \beta)^{-1} \cdot (1 - x_0 - (x_0 - \beta) - \beta^2)^{-1} \quad (x_0 < t < x_0 + \beta), \quad s(t) = 0 \quad (0 < t < x_0), \quad s(t) = 1/t \quad (x_0 + \beta < t < 1).
\]
Then \( r \) and \( s \) are continuous on \([0, 1]\).

From the Weierstrass theorem, we may choose polynomials \( p(x) \), \( P(x) \) such that
\[
0 < s(x) - p(x) < \varepsilon (1 + 2V(1))^{-1} \quad \text{and} \quad 0 < P(x) - r(x) < \varepsilon (1 + 2V(1))^{-1} \quad (0 \leq x \leq 1).
\]
Then, evidently, \( p(x) \leq s(x) \leq g(x) \) and \( g(x) \leq r(x) \leq P(x) \) \((0 \leq x \leq 1)\). Also,
\[
\int_{x_0}^{1} (r(t) - g(t)) \, d\sigma(t) = \int_{x_0 - \beta}^{x_0} r(t) \, d\sigma(t) \leq (V(x_0) - V(x_0 - \beta))/x_0 < \varepsilon/2.
\]
But
\[
\int_{x_0}^{1} P(t) \, d\sigma(t) < \varepsilon/2 + \int_{0}^{1} r(t) \, d\sigma(t),
\]
so
\[
\int_{0}^{1} P(t) \, d\sigma(t) < \varepsilon + \int_{0}^{1} g(t) \, d\sigma(t).
\]

Analogously, \( \int_{0}^{1} (g(t) - s(t)) \, d\sigma(t) \leq \int_{x_0}^{x_0 + \beta} (g(t) - s(t)) \, d\sigma(t) \leq \varepsilon (V(x_0 + \beta) - V(x_0))/x_0 < \varepsilon/2. \)

From \( \int_{0}^{1} s(t) \, d\sigma(t) < \varepsilon/2 + \int_{0}^{1} P(t) \, d\sigma(t) \), we obtain
\[
\int_{0}^{1} g(t) \, d\sigma(t) < \varepsilon + \int_{0}^{1} P(t) \, d\sigma(t).
\]

Fix \( m \in \{0, 1, 2, \ldots\} \). Then \( (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^{n+m} = (\varphi(x^{m+1})/\varphi(x)) \cdot ((\sum_{n=0}^{\infty} a_n (x^{m+1})^n)/\varphi(x^{m+1})) \) which tends to \( c_m \) when \( x \to 1 \), by hypothesis. But \( c_m = \int_{x_0}^{1} x^m \, d\sigma(x) \). Then, by linearity, \( \lim_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot Q(x^n) = \int_{0}^{1} Q(x) \, d\sigma(x) \) for all polynomials \( Q(x) \). Now, since \( g(x) \leq P(x) \) and \( \varphi(x) \) and the \( a_n \) \((n = 0, 1, 2, \ldots)\) are positive, we have \( \lim_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot g(x^n) \leq \lim_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot P(x^n) = \int_{0}^{1} P(t) \, d\sigma(t) < \varepsilon + \int_{0}^{1} g(t) \, d\sigma(t), \)
accordingly with (2). By doing \( \varepsilon \to 0 \), we obtain \( \lim \sup_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot g(x^n) \leq \int_{0}^{1} g(t) \, d\sigma(t) \). If we use \( p(x) \) instead of \( P(x) \), it follows similarly from (3) that \( \lim \inf_{x \to 1} (1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n x^n \cdot g(x^n) \geq \int_{0}^{1} g(t) \, d\sigma(t). \) Consequently,
\[
\lim_{x \to 1}(1/\varphi(x)) \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \cdot g(x^n) = \int_0^1 g(t) \, d\sigma(t) = \int_{x_0}^1 t^{-1} \, d\sigma(t) = \alpha. \]
Let \( x_j = x_0^{1/j} \) (\( j = 1, 2, \ldots \)). Then \( \sum_{n=0}^{\infty} a_n x_j^n \cdot g(x_j^n) = \sum_{n=0}^{j} a_n = S_j \), because
\[
g(x_j^n) = 1/x_j^n \quad \text{iff} \quad x_j^n \geq x_0 \quad \text{iff} \quad n \leq j
\]
\[
= 0 \quad \text{iff} \quad x_j^n < x_0 \quad \text{iff} \quad n > j.
\]
Finally, \( x_j \in (\delta, 1) \) for \( j \geq j_0 \) and \( \lim_{j \to \infty}(1/\varphi(x_j)) \). \( \sum_{n=0}^{\infty} a_n x_j^n \cdot g(x_j^n) = \alpha \).

Hence \( \lim_{j \to \infty} S_j/\varphi(x_0^{1/j}) = \alpha \). The theorem is proved.

It is clear that the Hardy-Littlewood theorem is a special case of this. In the classical theorem, \( c_m = 1/(m+1) \) and \( \sigma(x) \equiv x \). The point \( x_0 \in (0, 1) \) is arbitrary.

We give an easy example. Let \( \varphi(x) = \log((1-x)^{-1}) \). Then \( \varphi(x_0^{1/j}) \sim \log j \) (\( j \to \infty \)) for every \( x_0 \in (0, 1) \) and \( c_m = \lim_{x \to 1}(\log(1-x^{m+1}))/\log(1-x) = 1 \) (\( m = 0, 1, 2, \ldots \)). Then the sequence \( \{c_m\} \) is completely monotonic, because it satisfies (1) trivially. The function \( \sigma(x) = 0 \) (\( 0 \leq x < 1 \)), \( \sigma(1) = 1 \) is of bounded variation, continuous at every \( x_0 \in (0, 1) \) and satisfies \( c_m = \int_{x_0}^1 t^m \, d\sigma(t) \) (\( m = 0, 1, 2, \ldots \)). In addition, \( \alpha = \int_{x_0}^1 1/t \, d\sigma(t) = 1 \). Hence, we have: If \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) (\( 0 \leq x < 1 \)), \( a_n \geq 0 \) (\( n = 0, 1, 2, \ldots \)) and \( \lim_{x \to 1} f(x)/(\log(1-x)^{-1}) = 1 \), then \( S_n \sim \log n \) (\( n \to \infty \)).

**REMARK.** In [5, pp. 109–112], one can find several conditions which guarantee that we may choose \( \sigma \) to be continuous at each \( x_0 \in (0, 1) \). For the sake of completeness, we reproduce them:

(a) There exists a constant \( L \) such that
\[
(1+k) \cdot |\lambda_{k,m}| < L \quad \text{for} \quad (k, m = 0, 1, 2, \ldots; m \leq k).
\]

(b) There exist a number \( p > 1 \) and a constant \( L \) such that
\[
(1+k)^{p-1} \cdot \sum_{m=0}^{k} |\lambda_{k,m}|^p < L \quad \text{for} \quad (k = 0, 1, 2, \ldots).
\]

Here, \( \lambda_{k,m} = \binom{k}{m}(-1)^{k-m} \cdot \Delta^{k-m} c_m \).

**REFERENCES**


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