PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 4, December 1988

IS THERE A POINT OF ω^* THAT SEES ALL OTHERS?

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(Communicated by Thomas J. Jech)

ABSTRACT. If the cardinal c of the continuum is singular and p is an ultrafilter on ω of character c, then there is an ultrafilter q on ω which is not comparable to p in the Rudin-Keisler order.

1. Introduction. Beginning with the 1956 proof ([**6**, Theorem 1.5], attributed to H. Kenyon) that there are 2^c order-isomorphism classes of ultrafilters on ω, it has been clear that not all nonprincipal ultrafilters look alike. It is thus surprising that the following question (asked by van Mill in [**5**]) remains open: Is there a nonprincipal ultrafilter on ω which looks more or less like all of the others? That is, is there a nonprincipal ultrafilter on ω which is comparable to all others in the Rudin-Keisler order? We take the points of $\beta\omega$, the Stone-Čech compactification of the set ω of finite ordinals, to be the ultrafilters on ω, the principal ultrafilters being identified with points of ω. The points of $\omega^* = \beta\omega\backslash\omega$ are the nonprincipal ultrafilters. Given $p, q \in \beta\omega$ one says $p \leq q$ (in the Rudin-Keisler order) if and only if there exists $f\colon \omega \to \omega$ whose canonical extension f^β to $\beta\omega$ sends q to p. (See [**2**, Chapter 9, **5**, §3, or 1] for historical information and numerous results on the Rudin-Keisler order.) Given $f\colon \omega \to \omega$ and $p, q \in \beta\omega$ one easily sees that the statements $f^\beta(q) = p$, $(\forall A \in q)(f[A] \in p)$ and $(\forall A \in p)(f^{-1}[A] \in q)$ are equivalent. Thus, if $p \leq q$ one has that all of the structure of p is present in q. It is in this sense that p "looks more or less like" q. (It is known [**7**] that there is a subset of ω^* of size 2^c, no two members of which are comparable in the Rudin-Keisler order.)

It seems to the author intuitively clear that if the character of p, denoted by $\chi(p)$, is c (that is, if p is not generated by fewer than c sets), then there is some q which is not comparable to p. We prove this fact here under the additional assumption that c is a singular cardinal. (It is somewhat surprising that the singular case is easier to handle.)

The intuitive reason for adding the assumption that $\chi(p) = c$ can be appreciated by considering the extreme. If $\chi(p) = 1$, that is if p is principal, then for all $q \in \beta\omega$, $p \leq q$.

2. The results. The idea of the construction is simple enough. Given $p \in \omega^*$, there are only c functions from ω to ω which one writes as $\langle f_\alpha \rangle_{\alpha < c}$. One constructs q noncomparable to p by adding at stage α, A_α such that $f_\alpha[A_\alpha] \notin p$ (so $f_\alpha^\beta(q) \neq p$) and $f_\alpha^{-1}[A_\alpha] \notin p$ (so $f_\alpha^\beta(p) \neq q$). The only problem with the construction is that

Received by the editors December 3, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 04A20, 03E05; Secondary 54D35.

Key words and phrases. Ultrafilters, Rudin-Keisler order.

The author gratefully acknowledges support received from the National Science Foundation via Grant DMS-85-20873.

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one may finish too soon; the set $\{A_\sigma : \sigma < \alpha\}$ may generate an ultrafilter q with $q \leq p$. The following easy result summarizes when this simple construction works. (I know from conversation that Theorem 2.1 was obtained independently by van Mill.)

2.1 THEOREM. *Let $p \in \omega^*$ and assume that for all $r \leq p$, $\chi(r) = c$. Then there exists $q \in \omega^*$ such that q is not comparable to p.*

PROOF. Enumerate ${}^\omega\omega$ as $\langle f_\alpha \rangle_{\alpha < c}$. We inductively construct $\langle A_\alpha \rangle_{\alpha < c}$ so that for each α

(1) $f_\alpha[A_\alpha] \notin p$,

(2) $f_\alpha^{-1}[A_\alpha] \notin p$, and

(3) $\{A_\sigma : \sigma \leq \alpha\}$ has the uniform finite intersection property. (That is, if F is a finite subset of $\{\sigma : \sigma \leq \alpha\}$, then $|\bigcap_{\sigma \in F} A_\sigma| = \omega$.)

Let $\alpha < c$ and assume we have constructed $\langle A_\sigma \rangle_{\sigma < \alpha}$. Let $\mathscr{B} = \{A_\sigma : \sigma < \alpha\} \cup \{D \subseteq \omega : |\omega \backslash D| < \omega\}$. Then $\mathscr{B}$ has the uniform finite intersection property. We first produce C such that $f_\alpha^{-1}[C] \notin p$ and $\mathscr{B} \cup \{C\}$ has the uniform finite intersection property.

It may be that $\mathscr{B}$ generates an ultrafilter r. In this case $\chi(r) \leq |\mathscr{B}| < c$ so $r \nleq p$ so $f_\alpha^\beta(p) \neq r$. Pick $C \in r$ such that $f_\alpha^{-1}[C] \notin p$. Otherwise pick distinct r and s with $\mathscr{B} \subseteq r$ and $\mathscr{B} \subseteq s$. Pick $E \in r \backslash s$. Then $f_\alpha^{-1}[E] \cap f_\alpha^{-1}[\omega \backslash E] = \varnothing$ so pick $C \in \{E, \omega \backslash E\}$ with $f_\alpha^{-1}[C] \notin p$.

Now let $\mathscr{C} = \{\bigcap \mathscr{F} : \mathscr{F} \subseteq \mathscr{B} \bigcup \{C\}, \mathscr{F} \neq \varnothing$, and $|\mathscr{F}| < \omega\}$. Now $|\{f_\alpha[B] : B \in \mathscr{C}\}| < c$ so $\{f_\alpha[B] : B \in \mathscr{C}\}$ does not generate p. Pick $D \subseteq \omega$ such that $D \notin p$ and $\{f_\alpha[B] : B \in \mathscr{C}\} \cup \{D\}$ has the finite intersection property. It now suffices to show that $\mathscr{C} \cup \{f_\alpha^{-1}[D]\}$ has the finite intersection property. (For then, since $\mathscr{C}$ contains the cofinite sets, $\mathscr{C} \cup \{f_\alpha^{-1}[D]\}$ has the uniform finite intersection property. We then let $A_\alpha = C \cap f_\alpha^{-1}[D]$.)

Since $\mathscr{C}$ is closed under finite intersections, we let $B \in \mathscr{C}$ and show $B \cap f_\alpha^{-1}[D] \neq \varnothing$. But this follows immediately from the fact that $f_\alpha[B] \cap D \neq \varnothing$. □

2.2 COROLLARY. *If for all $p \in \omega^*$, $\chi(p) = c$ (in particular if Martin's Axiom, or just "$\mathfrak{p} = c$"), then for all $p \in \omega^*$ there exists $q \in \omega^*$ which is not comparable to p.*

The referee has suggested the following alternative proof of the "in particular" part of Corollary 2.2 which avoids the use of Theorem 2.1: It is well known that $\mathfrak{p} = c$ implies the existence of 2^c Ramsey ultrafilters. Any nonprincipal ultrafilter p is above at most c of these by definition of the Rudin-Keisler order, and is below no additional ones, since Ramsey ultrafilters are Rudin-Keisler minimal. So p is incomparable with most of the Ramsey ultrafilters.

2.3 THEOREM. *Assume $\mathrm{cf}(c) = \kappa < c$ and $p \in \omega^*$ with $\chi(p) = c$. Then there exists $q \in \omega^*$ which is not comparable to p.*

PROOF. Pick a sequence $\langle \gamma_\alpha \rangle_{\alpha < \kappa}$ which is cofinal in c and pick by **[4]** a $c \times c$ matrix of independent sets $\langle A_{\alpha,\delta} \rangle_{\alpha,\delta < c}$. That is, given $\alpha, \gamma, \delta < c$ with $\gamma \neq \delta$ one has $|A_{\alpha,\delta} \cap A_{\alpha,\gamma}| < \omega$ and given finite $F \subseteq c$ and $\mu : F \to c$, one has $|\bigcap_{\alpha \in F} A_{\alpha,\mu(\alpha)}| = \omega$.

Let $\mathscr{M} = \{f \in {}^\omega\omega :$ for all $x \in \omega$, $f^{-1}[\{x\}] \notin p\}$. We show first that

$$(*) \qquad \text{if } f \in \mathscr{M} \text{ and } \alpha < c, \text{ then } |\{\delta < c : f^{-1}[A_{\alpha,\delta}] \in p\}| \leq 1.$$

Indeed, given $\gamma < \delta < c$ one has $|A_{\alpha,\gamma} \cap A_{\alpha,\delta}| < \omega$ so if $f^{-1}[A_{\alpha,\delta} \cap A_{\alpha,\gamma}] \in p$, then $\bigcup_{x\in A_{\alpha,\delta}\cap A_{\alpha,\gamma}} f^{-1}[\{x\}] \in p$ so some $f^{-1}[\{x\}] \in p$, a contradiction.

Now enumerate $\mathscr{M}$ as $\langle f_\alpha\rangle_{\alpha<c}$. For each $\alpha < \kappa$, we choose $\mu(\alpha) < c$ such that for all $\sigma < \gamma_\alpha$, $f_\sigma^{-1}[A_{\alpha,\mu(\alpha)}] \notin p$. (This is possible by the observation (*).)

Let $\mathscr{A}_0 = \{A_{\alpha,\mu(\alpha)} : \alpha < \kappa\} \cup \{B \subseteq \omega : |\omega\backslash B| < \omega\}$. Enumerate ${}^\omega\omega$ as $\langle g_\sigma\rangle_{\sigma<c}$ with the identity as g_0. Let σ be given with $0 < \sigma < c$ and assume we have chosen $\{\mathscr{A}_\tau : \tau < \sigma\}$ so that, for each $\tau < \sigma$:

(1) $\mathscr{A}_\tau$ has the finite intersection property;

(2) if $\eta < \tau$, then $\mathscr{A}_\eta \subseteq \mathscr{A}_\tau$;

(3) $|\mathscr{A}_\tau| \le \max\{\kappa, |\tau|\}$; and

(4) there exists $D \in \mathscr{A}_\tau$ such that $g_\tau[D] \notin p$.

All hypotheses hold at $\tau = 0$. (For (4) observe that $g_0 \in \mathscr{M}$ and $g_0^{-1} = g_0$.) Let $\mathscr{B} = \{\bigcap \mathscr{F} : \mathscr{F} \subseteq \bigcup_{\tau<\sigma}\mathscr{A}_\tau,\ \mathscr{F} \neq \varnothing$, and $|\mathscr{F}| < \omega\}$. Now $|\bigcup_{\tau<\sigma}\mathscr{A}_\tau| \le \sum_{\tau<\sigma}|\mathscr{A}_\tau| \le \max\{\kappa,|\sigma|\}$ so $|\mathscr{B}| \le \max\{\kappa,|\sigma|\} < c$. Thus $|\{g_0[B] : B \in \mathscr{B}\}| < c$ so, since $\chi(p) = c$, pick $r \neq p$ in $\beta\omega$ such that $\{g_\sigma[B] : B \in \mathscr{B}\} \subseteq r$. Pick $C \in r\backslash p$ and let $\mathscr{A}_\sigma = \bigcup_{\tau<\sigma}\mathscr{A}_\tau \cup \{g_\sigma^{-1}[C]\}$. To see that $\mathscr{A}_\sigma$ has the finite intersection property, let $\mathscr{F}$ be a finite subset of $\bigcup_{\tau<\sigma}\mathscr{A}_\tau$. Then $\bigcap\mathscr{F} \in \mathscr{B}$ so $g_\sigma[\bigcap\mathscr{F}] \in r$ and hence $g_\sigma[\bigcap\mathscr{F}] \bigcap C \neq \varnothing$. Thus $\bigcap\mathscr{F} \bigcap g_\sigma^{-1}[C] \neq \varnothing$ as required.

The induction being complete, pick $q \in \beta\omega$ with $\bigcup_{\sigma<c}\mathscr{A}_\sigma \not\subseteq q$. Since $\{B \subseteq \omega : |\omega\backslash B| < \omega\} \subseteq \mathscr{A} \subseteq q$, we have $q \in \omega^*$. By hypothesis (4) we have $p \nleq q$.

To see that $q \nleq p$, let $h \in {}^\omega\omega$. If $h \in \mathscr{M}$, pick σ such that $h = f_\sigma$ and pick $\alpha < \kappa$ such that $\sigma < \gamma_\alpha$. Then $A_{\alpha,\mu(\alpha)} \in q$ while $h^{-1}[A_\alpha, \mu(\alpha)] \notin p$. Finally assume $h \notin \mathscr{M}$ and pick $x \in \omega$ such that $h^{-1}[\{x\}] \in p$. Then $h^\beta(p) = x \neq q$. □

A family $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is called an "independent family" provided, whenever $\mathscr{F}$ and $\mathscr{G}$ are disjoint finite subsets of $\mathscr{A}$, one has $|\bigcap\mathscr{F}\backslash\bigcup\mathscr{G}| = \omega$ (or, if $\mathscr{F} = \varnothing$, $|\omega\backslash\bigcup\mathscr{G}| = \omega$). It is reasonable, based on the above proof and several similar proofs in [**5**], to hope that given $p \in \omega^*$ and a suitably chosen independent family $\mathscr{A}$ one could choose for each $f \in {}^\omega\omega$ some $A \in \mathscr{A}$ with $f[A] \notin p$. Our last result shows that this is not possible.

2.4 THEOREM. *Let $\mathscr{A}$ be an independent family. Then there is some $f \in {}^\omega\omega$ such that $f[A]$ is cofinite for all $A \in \mathscr{A}$.*

PROOF. Consider first the possibility that $|\mathscr{A}| < \omega$. Then $|\bigcap\mathscr{A}| = \omega_{so}$ pick $f \in {}^\omega\omega$ with $f[\bigcap\mathscr{A}] = \omega$.

Thus $|\mathscr{A}| \ge \omega$ so choose a one-to-one sequence $\langle B_n\rangle_{n<\omega}$ in $\mathscr{A}$. Let $C_0 = \omega\backslash B_0$ and for $0 < n < \omega$, let $C_n = \bigcap_{t<n} B_t\backslash B_n$. Note that for $t < n$, $C_t \cap C_n = \varnothing$. Define $f \in {}^\omega\omega$ by $f[C_n] = \{n\}$ and (for definiteness) $f[\bigcap_{n<\omega} B_n] = \{0\}$. Let $A \in \mathscr{A}$.

If $A = B_t$ for some t, then for $n > t$ we have $C_n \subseteq A$ so that $f[A] \supseteq \{n \in \omega : n > t\}$.

We thus have that $A \notin \{B_n : n < \omega\}$. But this is better. We have $A \cap C_n \neq \varnothing$ for each n so that $f[A] = \omega$. □

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