WEAKLY NORMAL FILTERS AND THE CLOSED UNBOUNDED FILTER ON $P_\kappa \lambda$

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ABSTRACT. Assuming that $\kappa$ is supercompact and $\lambda$ is inaccessible, we present two isomorphic fine measures on $P_\kappa \lambda$ containing the closed unbounded filter. Some remarks on the (strongly) closed unbounded filter and weakly normal filters are added.

In the theory of $\kappa$-ultrafilters on a measurable cardinal $\kappa$, the closed unbounded filter (the club filter) plays an important role. For instance, Ketonen showed that any two distinct $\kappa$-ultrafilters containing the club filter are not isomorphic.

Weakly normal filters on a regular cardinal are also important. A filter is weakly normal iff it is a $p$-point containing the club filter. Every countably complete ultrafilter is minimal in the RK-ordering iff it is isomorphic to a weakly normal ultrafilter.

Jech is the first to introduce some combinatorial principles into $P_\kappa \lambda$ from the usual fields of $\kappa$. At first $P_\kappa \lambda$ seemed the same as $\kappa$. But it turned out to be more complicated. Menas proved that every normal measure on $P_\kappa \lambda$ where $\lambda$ is a strong limit with the cofinality less than $\kappa$ is isomorphic to a fine measure containing the club filter on $P_\kappa \lambda$. (See Proposition 12 in [9].) In [4], Gitik constructed a model in which there is a stationary subset of $P_\kappa \kappa^+$ that cannot be split into $\kappa^+$ disjointed stationary sets.

Applying Menas’ result, we present two isomorphic fine measures on $P_\kappa \lambda$ both of which contain the club filter under the hypothesis that $\kappa$ is supercompact and $\lambda$ is strongly inaccessible.

In [1], a kind of fine measure on $P_\kappa \lambda$ investigated by Menas, was studied. By the embedding argument, it was pointed out that such a measure is not normal and can be weakly normal in suitable conditions. We take a combinatorial approach and show that filters of the same type do not contain a standard club set, indeed strongly closed unbounded. We extend the results in [1] on the weak normality of such a filter.

At last, some remarks on the relation between the RK-order and weakly normal fine measures, the strongly club filter and the partition property are added.

0. Definitions and notations. $\kappa$ is a regular uncountable cardinal and $\lambda$ is a cardinal $> \kappa$ throughout. $P_\kappa \lambda = \{ x \subseteq \lambda : |x| < \kappa \}$. When we speak of a filter on $P_\kappa \lambda$ it is assumed to be $\kappa$-complete and fine, where $U$ is fine iff $\{ x : \alpha \in x \} \in U$ for all $\alpha < \lambda$. 

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DEFINITION 0.1. $U$ is normal if every regressive function is constant on a set of positive measure. (We write $X \in U^+$ if $X$ is positive measure.) $U$ is weakly normal if every regressive function is bounded by some $\gamma < \lambda$ on a set in $U$. We call $U$ a fine measure if it is an ultrafilter.

A subset $C$ of $P_\lambda \lambda$ is said to be unbounded if for each $a \in P_\lambda \lambda$ there is an $x \in C$ so that $a \subset x$. $\dot{a}$ denotes the set $\{x \in P_\lambda \lambda : a \subset x\}$. Thus $C$ is unbounded if $\dot{a} \cap C \neq \emptyset$ for all $a \in P_\lambda \lambda$. $C$ is closed if $\bigcup A \in C$ whenever $A$ is a $\subset$-increasing chain of length $< \kappa$ in $C$. $C$ is strongly closed if $\bigcup A \in C$ for all $A \subset C$ with $|A| < \kappa$. The club filter $\text{CF}_{\kappa \lambda}$ is the filter generated by the closed unbounded sets. The strongly club filter $\text{SCF}_{\kappa \lambda}$ is the filter generated by the strongly closed unbounded sets.

Let $U$ be a fine measure on $P_\lambda \lambda$ and $f : P_\lambda \rightarrow P_\lambda$. The ultrafilter $f_* (U)$ defined by "$X \in f_* (U)$ if $f^{-1} (X) \in U$" is a fine measure provided that $\{x : \alpha < f (x)\} \in U$ for all $\alpha < \lambda$.

DEFINITION 0.2. Suppose that $U$ and $D$ are fine measures on $P_\lambda \lambda$. We write $U \leq D$ if $U = f_* (D)$ for some $f : P_\lambda \lambda \rightarrow P_\lambda \lambda$. $U$ and $D$ are isomorphic ($U \cong D$) if $U = f_* (D)$ and $f$ is one-to-one on a set $X \in D$. $D$ is minimal in the RK-order if $D$ is isomorphic to all $U \leq D$.

DEFINITION 0.3. Suppose that $f$ is an ordinal valued function with domain $P_\lambda \lambda$. $f$ is the first function of $U$ if $\{x : f (x) > \gamma\} \in U$ for any $\gamma < \lambda$, and $\{x : g (x) < \gamma\} \in U$ for some $\gamma < \lambda$ whenever $\{x : g (x) < f (x)\} \in U$.

The first function tells us whether a fine measure is minimal or not under the certain assumption on $\lambda$.

DEFINITION 0.4. A fine measure $U$ has the partition property if every $F : [P_\lambda \lambda]^2 = \{(x, y) : x, y \in P_\lambda \lambda$ and $x \subseteq y\} \rightarrow 2$ has a homogeneous set in $U$. (A is homogeneous for $F$ if there is a $k < 2$ so that for all $x, y \in A$ with $x/ \subseteq y$, $F(\{x, y\}) = k$.)

1. Isomorphic fine measures. In this section, $\lambda$ is a fixed inaccessible cardinal greater than $\kappa$, a supercompact. We shall present two isomorphic fine measures including $\text{CF}_{\kappa \lambda}$. Though we extend the result of Menas, we have to start from it.

LEMMA 1.1 (Menas [9]). Let $\delta$ be a strong limit cardinal with the cofinality less than $\kappa$. Then every normal measure on $P_\delta \delta$ is isomorphic to a nonnormal fine measure containing $\text{CF}_{\kappa \delta}$.

Let $A = \{\delta : \kappa < \delta < \lambda, \delta$ is strong limit, $\text{cf}(\delta) < \kappa\}$. For each $\delta \in A$, there is a function $q^\delta : P_\delta \delta \rightarrow P_\kappa \delta$ so that $\text{CF}_{\kappa \delta} \subset q^\delta(\mathbb{U}_\delta) \equiv U_\delta$ where $U_\delta$ is a normal measure on $P_\kappa \delta$. We shall sum up these $U_\delta$'s and $q^\delta(\mathbb{U}_\delta)$'s with a suitable ultrafilter on $\lambda$.

LEMMA 1.2. There exists a $\kappa$-complete ultrafilter on $\lambda$ including $\{A\} \cup \text{CF}_\lambda$. ($\text{CF}_\lambda$ is the club filter on $\lambda$.)

PROOF. Since $\lambda$ is inaccessible, $A$ is stationary. Hence we have a $\lambda$-complete filter $E = \{X \subseteq \lambda : A - X$ is not stationary$\}$. It is easily seen that $\{A\} \cup \text{CF}_\lambda \subset E$. Then the strong compactness of $\kappa$ gives us a $\kappa$-complete ultrafilter $D$ extending $E$.

We use the above $D$. Define $F_1$ and $F_2$ by
\[ X \in F_1 \quad \text{if} \quad X \subseteq P_\lambda \lambda \text{ and } \{\delta \in A : X \cap P_\kappa \delta \in U_\delta\} \in D, \]
\[ X \in F_2 \quad \text{if} \quad X \subseteq P_\lambda \lambda \text{ and } \{\delta \in A : X \cap P_\kappa \delta \in q^\delta(\mathbb{U}_\delta)\} \in D. \]

$F_1$ and $F_2$ are fine measures on $P_\lambda \lambda$. We want to show that they are isomorphic and contain $\text{CF}_{\kappa \lambda}$. The next is an easy but key lemma.
**Lemma 1.3.** Assume that $\text{cf}(\eta) < \kappa$ and $U$ is a fine measure on $P_\kappa \eta$. Then 
\( \{ x \in P_\kappa \eta : \text{sup}(x) = \eta \} \in U. \)

**Proof.** Let \( \{ \eta_\alpha : \alpha < \text{cf}(\eta) \} \) be a cofinal subset of \( \eta \). Since \( U \) is fine, \( \{ x : \eta_\alpha \in x \} \in U \) for each \( \alpha < \text{cf}(\eta) \). Using the \( \kappa \)-completeness of \( U \) and the fact that \( \text{cf}(\eta) < \kappa \), we get \( \{ x : \eta_\alpha \in x \text{ for every } \alpha < \text{cf}(\eta) \} \in U. \)

**Corollary 1.4.** For every \( \delta \in A \), \( \{ x \in P_\kappa \delta : \text{sup}(x) = \delta \} \in U_\delta \) and \( \{ x \in P_\kappa \delta : \text{sup}(q^\delta(x)) = \delta \} \in U_\delta. \)

**Proof.** Since \( q^\delta(U_\delta) \) is also a fine measure on \( P_\kappa \delta \) and \( \text{cf}(\delta) < \kappa \), \( \{ x : \text{sup}(x) = \delta \} \in q^\delta(U_\delta) \). This is equivalent to \( \{ x : \text{sup}(q^\delta(x)) = \delta \} \in U_\delta. \)

For \( x \in P_\kappa \lambda \), let \( \delta_x \) be the least member of \( A \) such that \( x \in P_\kappa \delta \). And \( q : P_\kappa \lambda \to P_\kappa \lambda \) is defined by

\[
q(x) = q^\delta(x).
\]

By our construction,

**Lemma 1.5.** For every \( \delta \in A \), \( \{ x \in P_\kappa \delta : \delta_x = \delta \} \in U_\delta \); hence \( \{ x : q(x) = q^\delta(x) \} \in U_\delta. \)

We can see that \( F_1 \) and \( F_2 \) are isomorphic.

**Lemma 1.6.** \( q \) is one-to-one on a set in \( F_1 \).

**Proof.** Let \( B_\delta \in U_\delta \) be such that \( q^\delta \) is one-to-one on \( B_\delta \). We have already

known that \( C_\delta = \{ x \in B_\delta : q(x) = q^\delta(x), \text{sup}(q^\delta(x)) = \text{sup}(x) = \delta \} \in U_\delta \). Hence 
\( C = \bigcup \{ C_\delta : \delta \in A \} \) is a member of \( F_1 \).

Suppose that \( x, y \in C \) and \( q(x) = q(y) \). There is a \( \delta \in A \) such that \( \delta = \text{sup}(x) = \text{sup}(q(x)) = \text{sup}(y). \) Since \( x \) and \( y \) are in the same \( C_\delta \) and \( q \upharpoonright C_\delta = q^\delta \upharpoonright C_\delta \) is one-to-one, we have \( x = y \). Thus \( q \) is one-to-one on \( C \in F_1. \)

**Lemma 1.7.** \( F_2 = q^*(F_1). \)

**Proof.** Recall that \( X \in F_2 \) iff \( \{ \delta \in A : X \cap P_\kappa \delta \in q^\delta(U_\delta) \} \in D \), and that \( X \cap P_\kappa \delta \in q^\delta(U_\delta) \) is equivalent to \( \{ x \in P_\kappa \delta : q^\delta(x) \in X \cap P_\kappa \delta \} \in U_\delta \). By 1.5, the last paraphrase is the same as \( \{ x \in P_\kappa \delta : q(x) \in X \} \in U_\delta \).

Let \( Y = \{ x \in P_\kappa \lambda : q(x) \in X \} \). We have shown that \( X \in F_2 \) is equivalent to \( \{ \delta \in A : Y \cap P_\kappa \delta \in U_\delta \} \in D \). The latter says that \( Y \in F_1 \) and \( X \in q^*(F_1). \) Hence \( X \in F_2 \) iff \( X \in q^*(F_1). \)

What is left to show is that both \( F_1 \) and \( F_2 \) contain \( \text{CF}_{\kappa \lambda} \). Note that \( \{ \delta < \lambda : X \cap P_\kappa \delta \in \text{CF}_{\kappa \delta} \} \in \text{CF}_{\kappa \lambda} \) for every \( X \in \text{CF}_{\kappa \lambda}. \)

**Lemma 1.8.** \( \text{CF}_{\kappa \lambda} \subseteq F_1 \cap F_2. \)

**Proof.** Suppose that \( X \in \text{CF}_{\kappa \lambda} \). Then \( X' = \{ \delta < \lambda : X \cap P_\kappa \delta \in \text{CF}_{\kappa \delta} \} \in \text{CF}_{\kappa \lambda} \subseteq D \). Since \( U_\delta \) and \( q^\delta(U_\delta) \) contain \( \text{CF}_{\kappa \delta} \), \( X \cap P_\kappa \delta \) belongs to both \( U_\delta \) and \( q^\delta(U_\delta) \) for all \( \delta \in X'. \) Hence \( X \in F_1 \cap F_2. \)

Now we are done.

**Theorem 1.9.** If \( \lambda \) is a strongly inaccessible cardinal greater than \( \kappa \) a supercompact, there are two distinct isomorphic fine measures on \( P_\kappa \lambda \) containing the club filter.

The author does not know whether a normal measure on \( P_\kappa \lambda \) is isomorphic to a fine measure containing \( \text{CF}_{\kappa \lambda} \) under the same assumption. It is also still open...
whether two fine measures can be isomorphic for a successor cardinal \( \lambda \). The case that \( \lambda \) is not strong limit is also open.

2. SCF\( _{\kappa \lambda} \), prestationary sets and the partition property. For the subsets of regular uncountable cardinals, the situation is simple. That is, \( S \subseteq \kappa \) is stationary iff for any regressive function \( f \) on \( S \), there is an unbounded set \( T \subseteq S \) on which \( f \) is constant. But this does not hold for the subsets of \( P_{\kappa \lambda} \).

In this section, \( \kappa \) is a regular uncountable cardinal and \( \lambda > \kappa \). We begin by Menas’ invention again.

**Proposition 2.1 (Menas [8]).** There is a nonstationary subset \( S \) of \( P_{\kappa \lambda} \) such that every regressive function is constant on an unbounded subset of \( S \).

**Definition 2.2.** We call such a set \( S \) “prestationary”.

Menas characterized \( S \) “stationary” as follows:

**Proposition 2.3 (Menas [8]).** \( S \subseteq P_{\kappa \lambda} \) is stationary iff any function \( f : S \to \lambda \times \lambda \) so that \( f(y) \in y \times y \) for all \( y \in S \), is constant on some unbounded \( T \subseteq S \).

In the spirit of Proposition 2.3, we can express stationarity using prestationarity.

**Proposition 2.4.** If \( S \subseteq P_{\kappa \lambda} \) is prestationary and every regressive function is constant on a prestationary \( T \subseteq S \), then \( S \) is stationary.

**Proof.** Let \( f : S \to \lambda \times \lambda \), \( f_1, f_2 : S \to \lambda \) so that \( f(y) \in y \times y \) for all \( y \in S \) and \( f(y) = (f_1(y), f_2(y)) \). Since \( f_1(y) \in y \) for all \( y \in S \), there is a prestationary \( T_1 \subseteq S \) on which \( f_1 \) is constant. Again by the fact that \( f_2(y) \in y \) for every \( y \) in \( T_1 \) that is prestationary, there is an unbounded \( T_2 \subseteq T_1 \) so that \( f_2 \upharpoonright T_2 \) is constant. Then \( f \upharpoonright T_2 \) is constant. \( \square \)

The stationary subsets are the sets which have nonempty intersection with every closed unbounded set. Now we characterize the prestationary sets with SCF\( _{\kappa \lambda} \). First recall the theorem for SCF\( _{\kappa \lambda} \) in Carr [3].

**Lemma 2.5 (Carr).** \( C \subseteq SCF_{\kappa \lambda} \) iff there is a sequence of sets in \( P_{\kappa \lambda} \), \( \langle x_\alpha | \alpha < \lambda \rangle \) so that \( \Delta(\dot{x}_\alpha | \alpha < \lambda) = \{ y : x_\alpha \subseteq y \text{ for all } \alpha \in y \} \subseteq C \).

**Proposition 2.6.** \( S \subseteq P_{\kappa \lambda} \) is prestationary iff \( S \cap C \neq 0 \) for all \( C \subseteq SCF_{\kappa \lambda} \).

**Proof.** Suppose that \( S \) is prestationary and \( S \cap C = 0 \) for some \( C \subseteq SCF_{\kappa \lambda} \). By 2.5, there is a sequence \( \langle x_\alpha | \alpha < \lambda \rangle \) so that \( \Delta(\dot{x}_\alpha | \alpha < \lambda) \subseteq C \). For every \( x \in S \), there exists an \( \alpha \in x \) such that \( x_\alpha \not\subseteq x \). Since \( S \) is prestationary, there is an ordinal \( \gamma \) so that \( \{ x \in S : x_\gamma \not\subseteq x \} \) is unbounded. Contradiction.

For the converse, assume that \( S \cap C \neq 0 \) for all \( C \subseteq SCF_{\kappa \lambda} \) and \( S \) is not prestationary. There is a regressive function \( f \) such that for every \( \alpha < \lambda \) there is an \( a_\alpha \in P_{\kappa \lambda} \) so that \( \{ x \in S : f(x) = \alpha \} \cap a_\alpha = 0 \). Let \( C = \Delta(a_\alpha | \alpha < \lambda) \); then \( C \subseteq SCF_{\kappa \lambda} \). Pick an \( x \in C \cap S \) and suppose that \( f(x) = \alpha \). Since \( \alpha \in x \) and \( x \in C \), \( a_\alpha \subseteq x \). Then \( f(x) \neq \alpha \) by the definition of \( a_\alpha \). This is absurd. \( \square \)

We connect the above fact to the partition property of fine measures.

**Corollary 2.7.** If \( U \) is a fine measure with the partition property assigning measure one to the strongly club sets, then \( U \) is normal.

This is really Proposition 11 in Menas [9], where he proved it for the club sets version. Menas’ proof is applicable in our case as well.
3. Weakly normal filters on $P_\kappa\lambda$. For weakly normal filters on $\kappa$ regular, see Kanamori [7]. We briefly review the basic facts.

PROPOSITION 3.1. For any filter on $\kappa$, the following are equivalent.

(i) $U$ is weakly normal.

(ii) Every filter extension of $U$ is weakly normal.

(iii) If $\{X_\alpha : \alpha < \kappa\}$ are sets of positive measure such that $X_\beta \subseteq X_\alpha$ whenever $\alpha < \beta$, then $\Delta\{X_\alpha : \alpha < \kappa\} = \{\alpha < \kappa : \alpha \in \beta \text{ for all } \beta < \alpha\}$ has a positive measure.

(iv) $U$ is a $p$-point filter extending $\text{CF}_\kappa$. ($U$ is a $p$-point if every function $f : \kappa \rightarrow \kappa$ such that $\kappa - f^{-1}(\{\alpha\}) \subseteq U$ for all $\alpha < \kappa$ is $< \kappa$ to one on some $X \in U$.)

It is natural to ask whether the same thing happens to filters on $P_\kappa\lambda$. We easily get that (i)$\sim$(iii) are also equivalent for any filter on $P_\kappa\lambda$. (Note that $\Delta\{X_\alpha : \alpha < \lambda\} = \{x \in P_\kappa\lambda : x \in X_\alpha \text{ for all } \alpha \in X\}$.)

But for (iv), the author only knows the following.

PROPOSITION 3.2. (i) Suppose that $U$ is weakly normal. If $f$ is a function with the domain $P_\kappa\lambda$ and $\{x : f(x) > \alpha\} \subseteq U^+$ for all $\alpha < \lambda$, then there is a set $X$ of positive measure so that $X \cap f^{-1}(\{\alpha\}) \subseteq P_\kappa\lambda$ for all $\alpha < \lambda$.

(ii) Suppose that $U$ extends $\text{SCF}_{\kappa\lambda}$ and for any $\alpha < \lambda$ there is an $X \in U^+$ such that $X \cap f^{-1}(\{\alpha\}) \subseteq P_\kappa\beta$ for some $\beta < \lambda$ whenever $f$ satisfies $\{x : f(x) > \gamma\} \subseteq U^+$ for some $\gamma < \lambda$. Then $U$ is weakly normal.

PROOF. (i) Let $X_\xi = \{x : f(x) > \xi\}$ for each $\xi < \lambda$. Then $X_\xi \subseteq U^+$ and $X_\eta \subseteq X_\xi$ if $\xi < \eta$. Now $\Delta\{X_\xi : \xi < \lambda\} \subseteq U^+$ by (iii). If $x \in \Delta\{X_\xi : \xi < \lambda\}$ and $f(x) = \alpha$, then $\xi < \alpha$ for all $\xi \in x$. Hence $x \subset\alpha$.

(ii) Suppose that $f$ is a regressive function on $P_\kappa\lambda$. Since $U$ extends $\text{SCF}_{\kappa\lambda}$, every $X$ of positive measure is prestationary. Hence there is an $\alpha < \lambda$ so that $X \cap f^{-1}(\{\alpha\})$ is unbounded. By our hypothesis, $\{x : f(x) < \gamma\} \subseteq U^+$ for some $\gamma < \lambda$. The question left is whether every weakly normal filter extends $\text{CF}_{\kappa\lambda}$ or $\text{SCF}_{\kappa\lambda}$. In [1], the fine measure investigated by Menas was revisited and shown to be non-normal. We again observe it and get more information, which gives a negative answer to the question. The author wishes to express his gratitude to A. Blass whose advice led to a simplified proof. We concentrate on a filter defined below.

We assume that $\kappa$ is a regular limit cardinal.

Let $\langle U_\alpha : \alpha < \kappa \rangle$ be a sequence of fine filters on $P_\alpha\lambda$ and $D$ be a $\kappa$-complete uniform filter on $\kappa$. Then a fine filter $U$ is defined by $X \in U$ if $X \subseteq P_\kappa\lambda$ and $\{\alpha < \kappa : X \cap P_\alpha\lambda \subseteq U_\alpha\} \subseteq D$.

THEOREM 3.3 (INSPIRED BY BLASS). $U$ does not extend $\text{SCF}_{\kappa\lambda}$ hence is nonnormal.

PROOF. Let $C = \{x \in P_\kappa\lambda : x \cap \kappa \text{ is an ordinal}\}$. Then $C$ is strongly closed unbounded. We shall show that $C \cap P_\alpha\lambda$ is not unbounded for all $\alpha < \kappa$. If $x \in C \cap P_\alpha\lambda$ and $\alpha^+ \subseteq x$, then $\alpha^+ \subseteq x$. But this contradicts $|x| < \alpha$. Hence $\alpha^+ \not\subseteq x$ for all $x \in C \cap P_\alpha\lambda$. Thus $C \not\subseteq U$. Note that $\alpha^+ < \kappa < \lambda$ since $\kappa$ is a limit cardinal. □

For certain $A \subseteq \kappa$ we have a strongly club set which is not unbounded for any $\alpha \in A$. More precisely;
PROPOSITION 3.4. Suppose that $\lambda^\kappa = \lambda$ and $A \subseteq \kappa$. There is a $C \in \text{SCF}_{\kappa\lambda}$ so that if $\alpha \in A$ and $\sup(A \cap \alpha) \neq \alpha$, then $C \cap P_\alpha \lambda$ is not unbounded.

PROOF. Let $\{x_\xi : \xi < \lambda\}$ be an enumeration of $P_\lambda \lambda$ and $\alpha_\xi = \text{the least member of } A > |x_\xi|$. Then, we pick a $y_\xi \supset x$ with $|y_\xi| \geq \alpha_\xi^+$. Finally, $C = \Delta(y_\xi : \xi < \lambda)$.

Suppose that $\alpha \in A$ and $\sup(A \cap \alpha) \neq \alpha$. Then $\alpha = \alpha_\xi$ for some $x_\xi$. Assume that there exists an $x \in C \cap P_\lambda \lambda$ with $\xi \in x$. By our definition of $C$, $x \supset y_\xi$. This implies $|x| \geq |y_\xi| \geq \alpha_\xi^+ > \alpha$ contradicting $x \in P_\alpha \lambda$. Hence $(C \cap P_\alpha \lambda) \cap \{\xi\} = 0$. □

Now we turn to the weak normality of $U$ under the assumption that $U_\alpha$ is weakly normal for all $\alpha < \kappa$, and improve Proposition 2.4 in [1] by a simple argument. In the next theorem, $\kappa$ is not necessarily a limit cardinal in (i) and (iii).

THEOREM 3.5. (i) $\text{cf}(\lambda) > \kappa$, then $U$ is weakly normal.
(ii) $\text{cf}(\lambda) = \kappa$, then $U$ is not weakly normal.
(iii) If $\text{cf}(\lambda) < \kappa$ and (a) or (b) is satisfied, then $U$ is weakly normal.

(a) $U$ is an ultrafilter.
(b) $D$ is $\text{cf}(\lambda)$-descendingly complete. That is, if $\{X_\xi : \xi < \text{cf}(\lambda)\}$ is a sequence of positive measure such that $X_\eta \subset X_\xi$ whenever $\xi < \eta$, then $\bigcap\{X_\xi : \xi \in \text{cf}(\lambda)\} \neq 0$.

(Note that $D$ is not required to be an ultrafilter.)

PROOF. Suppose that $f(x) \in x$ for every $x \in P_\lambda \lambda$.

(i) For $\alpha < \kappa$, $\delta_\alpha$ is an ordinal $< \lambda$ such that $\{x \in P_\alpha \lambda : f(x) < \delta_\alpha\} \in U_\alpha$. Since $\text{cf}(\lambda) > \kappa$, $\delta = \sup(\{\delta_\alpha : \alpha < \kappa\}) < \lambda$. Obviously $\{x \in P_\kappa \lambda : f(x) < \delta\} \in U$.

(ii) Let $\{\lambda_\alpha : \alpha < \kappa\}$ be a cofinal subset of $\lambda$ and $\lambda_\alpha < \lambda_\beta$ if $\alpha < \beta$. For each $\alpha < \kappa$, $\{x \in P_\alpha \lambda : \lambda_\alpha \in x \text{ and } \lambda_\alpha < \lambda_\alpha \} \in U$. Hence we have $\{x \in P_\kappa \lambda : x - \lambda_{|x|} \neq 0\} \in U$.

So, there is a function $g : P_\kappa \lambda \rightarrow \lambda$ such that $g(x) \in x$ and $g(x) > \lambda_{|x|}$ for almost all $x$ (mod $U$). For any $\alpha < \kappa$, we know that $\{x \in P_\alpha \lambda : x \supset \alpha^+\} \in U$ and then $\{x : \lambda_{|x|} > \lambda_\alpha\} \in U$. Hence $\{x \in P_\kappa \lambda : g(x) > \lambda_\alpha\} \in U$ for every $\alpha < \kappa$. We are done because $g$ is an unbounded regressive function.

(iii) Suppose that (a) holds. We already showed in Lemma 1.3 that every fine measure on $P_\lambda \lambda$ is weakly normal if $\text{cf}(\lambda) < \kappa$. In fact,

Fact 3.6. A fine measure is weakly normal iff its first function maps $x$ to $\sup(x)$. (We denote such a function by $\text{Sup}$.)

When (b) holds, let $\{\lambda_\alpha : \alpha < \delta\}$ be a cofinal subset of $\lambda$ with $\delta = \text{cf}(\lambda)$ so that $\lambda_\alpha < \lambda_\beta$ if $\alpha < \beta$. Suppose that $\{x \in P_\kappa \lambda : f(x) < \lambda_\alpha\} \notin U$ for all $\alpha < \delta$. Then $\{\xi < \kappa : \{x \in P_\xi \lambda : f(x) < \lambda_\alpha\} \notin U_\xi\} \notin D$ for any $\alpha < \delta$. Hence

$C_\alpha = \{\xi < \kappa : \{x \in P_\xi \lambda : f(x) < \lambda_\alpha\} \notin U_\xi\} \in D^+$.

If $\alpha < \beta$, then $\{x \in P_\beta \lambda : f(x) < \lambda_\beta\} \notin U_\xi$ implies $\{x \in P_\beta \lambda : f(x) < \lambda_\alpha\} \notin U_\xi$ since $\lambda_\alpha < \lambda_\beta$. So, $C_\delta \subset C_\alpha$. Then $C = \bigcap\{C_\alpha : \alpha < \delta\} \neq 0$.

Pick a $\xi \in C$. $\{x \in P_\xi \lambda : f(x) < \lambda_\alpha\} \notin U_\xi$ for any $\alpha < \delta$. This contradicts the hypothesis that $U_\xi$ is weakly normal. □

Note that a filter $F$ on $P_\lambda \lambda$ is weakly normal if it is $\text{cf}(\lambda)$-descendingly complete. Combining Theorems 3.3 and 3.5, we have;

COROLLARY 3.7. There is a weakly normal filter which does not extend $\text{SCF}_{\kappa\lambda}$.

Jech [5] and Carr [3] showed that $\text{CF}_{\kappa\lambda}$ is the minimal normal filter. Is there a nice analogue for weakly normal filter? Or, what is the consistency of weakly normal filters? (Note here we assume that any filter is fine and $\kappa$-complete.)
4. Weakly normal fine measures and the RK-ordering. In this section, \( \kappa \) is a fixed strongly compact cardinal. We observe the weak normality in view of the RK-ordering. First we review the fact established by Menas in [8].

**Theorem 4.1 (Menas).** (i) If \( \text{cf}(\lambda) < \kappa \) or \( \lambda \) is regular, then every normal measure on \( P_\kappa \lambda \) is minimal.

(ii) If \( \lambda \) is regular and the first function of \( U \) is one-to-one on a set of measure one, then \( U \) is minimal.

We hope that every weakly normal measure is minimal as in the theory of uniform ultrafilters on a regular cardinal. In fact any minimal fine measure is isomorphic to a weakly normal measure.

**Proposition 4.2.** Every fine measure has a weakly normal measure below it.

**Proof.** Let \( U \) be a fine measure and \( g \) its first function. Define \( f: P_\kappa \lambda \to P_\kappa \lambda \) by \( f(x) = x \cap g(x) \).

By an easy observation, \( \{x: \alpha \in f(x)\} \in U \) for all \( \alpha < \lambda \) and \( f_* (U) \) is a fine measure.

Suppose that \( \{x: f(x) \in x\} \in f_* (U) \). It means that \( \{x: h \circ f(x) \in x \cap g(x)\} \in U \).

Since \( g \) is the first function of \( U \), we have \( \{x: h \circ f(x) < \gamma\} \in U \) for some \( \gamma < \lambda \).

Hence \( \{x: h(x) < \gamma\} \in f_* (U) \). □

The next fact appeared already in [8] implicitly.

**Proposition 4.3.** Let \( \lambda \) be regular and \( U \) a fine measure on \( P_\kappa \lambda \). \( U \) is minimal iff its first function is one-to-one on a set \( X \in U \).

**Proof.** Let \( \{A_\lambda (\alpha): \alpha < \lambda\} \) be a partition of \( \{\alpha < \lambda: \text{cf}(\alpha) = \omega\} \) into disjointed stationary subsets. Let \( f \) be the first function and define \( q \) by \( q(x) = \{\alpha < f(x): A_\lambda (\alpha) \cap f(x) \) is stationary in \( f(x)\} \). Then \( q_* (U) \) is a minimal fine measure (Theorem 2.14 in [8]).

Suppose that \( U \) is minimal. \( q \upharpoonright X \) is one-to-one for some \( X \in U \). But \( q(x) = q(y) \) if \( f(x) = f(y) \). Hence \( f \upharpoonright X \) is one-to-one. □

**Corollary 4.4.** A weakly normal measure on \( P_\kappa \lambda \) with \( \lambda \) regular is minimal iff \( \sup \) is one-to-one on a set of measure one.

A filter \( F \) on a regular cardinal \( \rho \) is called a \( q \)-point if every \( < \rho \) to one function from \( \rho \) to \( \rho \) is one-to-one on a set \( X \in F \). It is known that any filter extending \( CF_\rho \) is a \( q \)-point. \( SCF_{\kappa \lambda} \) also plays a role on the minimality of weakly normal measures.

**Proposition 4.5.** Let \( \lambda \) be regular. If \( U \) is a minimal fine measure on \( P_\kappa \lambda \) that is not weakly normal, then \( SCF_{\kappa \lambda} \not\in U \).

**Proof.** Let \( f \) be the first function. By our assumption, there is a set \( X \in U \) so that \( f \upharpoonright X \) is one-to-one and \( f(x) < \sup(x) \) for all \( x \in X \).

Suppose that \( SCF_{\kappa \lambda} \subset U \). Then \( X \) is prestationary. For \( x \in X \), set \( g(x) = \) the least member of \( x \) greater than \( f(x) \). There is an unbounded set \( Y \subset X \) such that \( g^\gamma Y = \{\gamma\} \) for some \( \gamma < \lambda \). Thus, \( f'' Y \subset \gamma \) and \( |Y| = \lambda^{< \kappa} > \gamma \), which contradicts the fact that \( f \upharpoonright Y \) is one-to-one. □
COROLLARY 4.6. Let $\lambda$ be regular. If $U$ is normal and $f_*(U) \supseteq \text{SCF}_{\kappa\lambda}$, then $f_*(U)$ is weakly normal and $\{x: \sup(f(x)) = \sup(x)\} \in U$.

COROLLARY 4.7. For any regular $\lambda > \kappa$, there is a nonminimal fine measure extending $\text{CF}_{\kappa\lambda}$.

PROOF. Let $A = \{\alpha < \lambda: \text{cf}(\alpha) < \kappa\}$ which is stationary in $\lambda$. We repeat the construction in §1.

There is a $\kappa$-complete ultrafilter on $\lambda$, $D \supseteq \text{CF}_\lambda \cup \{A\}$. For each $\alpha \in A$, fix a fine filter $U_\alpha$ on $P_\kappa\alpha$ extending $\text{CF}_{\kappa\alpha}$, and define $U$ by

$$X \in U \iff \{\alpha < \lambda: X \cap P_\kappa\alpha \in U_\alpha\} \in D.$$  

Then $U$ is a fine measure extending $\text{CF}_{\kappa\lambda}$.

We shall see that $U$ is not weakly normal, hence nonminimal by Proposition 4.5.

Since $A \in D$, $D$ is not normal. Thus there is a function $g$ so that $[g]_D = \lambda$ and $\{\alpha < \lambda: g(\alpha) < \alpha\} \in D$.

For $x \in P_\kappa\lambda$, let $\alpha_x = \text{the least } \alpha \text{ such that } x \in P_\kappa\alpha$ and $f(x) = g(\alpha_x)$. For every $\alpha \in A$, $\{x \in P_\kappa\alpha: f(x) < \sup(x)\} \in U_\alpha$ since $\{x: \alpha_x = \alpha = \sup(x)\} \in U_\alpha$. Let $h(x) = \text{the least member of } x \text{ greater than } f(x)$. $h$ is a regressive function on a set in $U$.

Pick a $\gamma < \lambda$. Then $B = \{\alpha \in A: \gamma < g(\alpha)\} \in D$. For all $\alpha \in B$, $\{x \in P_\kappa\alpha: f(x) = g(\alpha_x) = g(\alpha)\} \in U_\alpha$ and $\{x \in P_\kappa\alpha: f(x) > \gamma\} \in U_\alpha$. Hence $\{x \in P_\kappa\lambda: \gamma < f(x)\} \in U$. It shows that Sup is not the least function. $\Box$

On the other hand, we have a minimal fine measure which is weakly normal and does not extend $\text{SCF}_{\kappa\lambda}$. We recall the fine measure in §3. Suppose that $\lambda$ is regular and $\langle U_\alpha|\alpha < \kappa\rangle$ is a sequence of normal measures on $P_\kappa\lambda$ and $D$ is a normal measure on $\kappa$. Define $U$ by

$$X \in U \iff \{\alpha < \kappa: X \cap P_\alpha\lambda \in U_\alpha\} \in D.$$  

Following the argument of 3.1, 3, 4 in [10], we get

LEMMA 4.8. (i) $\{x: \text{the order type of } x \text{ is regular}\} \subseteq U$.

(ii) Let $G$ be a $\omega$-Jonsson function over $\lambda$. ($G$ is $\omega$-Jonsson over $y$ if $G: \omega \rightarrow y$ and $G^nz = y$ whenever $z \subseteq y$ and $|z| = |y|$.) Then we have $\{x: G \upharpoonright \omega x \text{ is } \omega \text{-Jonsson over } x\} \subseteq U$.

(iii) There is an $X \in U$ so that $\text{Sup} \upharpoonright X$ is one-to-one.

Note that normality of $U_\alpha$'s is necessary in the above. Using the results proved in §3, we can show

THEOREM 4.9. For every regular $\lambda > \kappa$, there is a weakly normal minimal fine measure which does not extend $\text{SCF}_{\kappa\lambda}$.

PROOF. It is clear that every normal measure is weakly normal. Hence our $U$ is weakly normal by Theorem 3.5(i). Theorem 3.3 asserts that $U$ does not extend $\text{SCF}_{\kappa\lambda}$. At last $U$ is minimal by Fact 3.6, Theorem 4.1(ii), and Lemma 4.8(iii). $\Box$

It is not known whether $U$ can be isomorphic to some fine measure extending $\text{SCF}_{\kappa\lambda}$. We also do not know whether nonminimal weakly normal measures exist.
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