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WEAKLY NORMAL FILTERS AND THE CLOSED UNBOUNDED FILTER ON $P_\kappa\lambda$

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ABSTRACT. Assuming that κ is supercompact and λ is inaccessible, we present two isomorphic fine measures on $P_\kappa\lambda$ containing the closed unbounded filter. Some remarks on the (strongly) closed unbounded filter and weakly normal filters are added.

In the theory of κ-ultrafilters on a measurable cardinal κ, the closed unbounded filter (the club filter) plays an important role. For instance, Ketonen showed that any two distinct κ-ultrafilters containing the club filter are not isomorphic.

Weakly normal filters on a regular cardinal are also important. A filter is weakly normal iff it is a p-point containing the club filter. Every countably complete ultrafilter is minimal in the RK-ordering iff it is isomorphic to a weakly normal ultrafilter.

Jech is the first to introduce some combinatorial principles into $P_\kappa\lambda$ from the usual fields of κ. At first $P_\kappa\lambda$ seemed the same as κ. But it turned out to be more complicated. Menas proved that every normal measure on $P_\kappa\lambda$ where λ is a strong limit with the cofinality less than κ is isomorphic to a fine measure containing the club filter on $P_\kappa\lambda$. (See Proposition 12 in [**9**].) In [**4**], Gitik constructed a model in which there is a stationary subset of $P_\kappa\kappa^+$ that cannot be split into κ^+ disjointed stationary sets.

Applying Menas' result, we present two isomoprhic fine measures on $P_\kappa\lambda$ both of which contain the club filter under the hypothesis that κ is supercompact and λ is strongly inaccessible.

In [**1**], a kind of fine measure on $P_\kappa\lambda$ investigated by Menas, was studied. By the embedding argument, it was pointed out that such a measure is not normal and can be weakly normal in suitable conditions. We take a combinatorial approach and show that filters of the same type do not contain a standard club set, indeed strongly closed unbounded. We extend the results in [**1**] on the weak normality of such a filter.

At last, some remarks on the relation between the RK-order and weakly normal fine measures, the strongly club filter and the partition property are added.

0. Definitions and notations. κ is a regular uncountable cardinal and λ is a cardinal $> \kappa$ throughout. $P_\kappa\lambda = \{x \subset \lambda : |x| < \kappa\}$. When we speak of a filter on $P_\kappa\lambda$ it is assumed to be κ-complete and fine, where U is fine iff $\{x\colon \alpha \in x\} \in U$ for all $\alpha < \lambda$.

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DEFINITION 0.1. U is normal if every regressive function is constant on a set of positive measure. (We write $X \in U^+$ if X is positive measure.) U is weakly normal if every regressive function is bounded by some $\gamma < \lambda$ on a set in U. We call U a fine measure if it is an ultrafilter.

A subset C of $P_\kappa\lambda$ is said to be unbounded if for each $a \in P_\kappa\lambda$ there is an $x \in C$ so that $a \subset x$. $\hat{a}$ denotes the set $\{x \in P_\kappa\lambda : a \subset x\}$. Thus C is unbounded if $\hat{a} \cap C \neq 0$ for all $a \in P_\kappa\lambda$. C is closed if $\bigcup A \in C$ whenever A is a $\subset$-increasing chain of length $< \kappa$ in C. C is strongly closed if $\bigcup A \in C$ for all $A \subset C$ with $|A| < \kappa$. The club filter $\mathrm{CF}_{\kappa\lambda}$ is the filter generated by the closed unbounded sets. The strongly club filter $\mathrm{SCF}_{\kappa\lambda}$ is the filter generated by the strongly closed unbounded sets.

Let U be a fine measure on $P_\kappa\lambda$ and $f : P_\kappa\lambda \to P_\kappa\lambda$. The ultrafilter $f_*(U)$ defined by "$X \in f_*(U)$ if $f^{-1}(X) \in U$" is a fine measure provided that $\{x : \alpha \in f(x)\} \in U$ for all $\alpha < \lambda$.

DEFINITION 0.2. Suppose that U and D are fine measures on $P_\kappa\lambda$. We write $U \leq D$ if $U = f_*(D)$ for some $f : P_\kappa\lambda \to P_\kappa\lambda$. U and D are isomorphic ($U \cong D$) if $U = f_*(D)$ and f is one-to-one on a set $X \in D$. D is minimal in the RK-order if D is isomorphic to all $U \leq D$.

DEFINTION 0.3. Suppose that f is an ordinal valued function with domain $P_\kappa\lambda$. f is the first function of U if $\{x : f(x) > \gamma\} \in U$ for any $\gamma < \lambda$, and $\{x : g(x) < \gamma\} \in U$ for some $\gamma < \lambda$ whenever $\{x : g(x) < f(x)\} \in U$.

The first function tells us whether a fine measure is minimal or not under the certain assumption on λ.

DEFINITION 0.4. A fine measure U has the partition property if every $F : [P_\kappa\lambda]^2 = \{\{x, y\} : x, y \in P_\kappa\lambda \text{ and } x \subsetneq y\} \to 2$ has a homogeneous set in U. (A is homogeneous for F if there is a $k < 2$ so that for all $x, y \in A$ with $x / \subsetneq y$, $F(\{x, y\}) = k$.)

1. Isomorphic fine measures. In this section, λ is a fixed inaccessible cardinal greater than κ, a supercompact. We shall present two isomorphic fine measures including $\mathrm{CF}_{\kappa\lambda}$. Though we extend the result of Menas, we have to start from it.

LEMMA 1.1 (MENAS [**9**]). *Let δ be a strong limit cardinal with the cofinality less than κ. Then every normal measure on $P_\kappa\delta$ is isomorphic to a nonnormal fine measure containing* $\mathrm{CF}_{\kappa\delta}$.

Let $A = \{\delta : \kappa < \delta < \lambda,\ \delta \text{ is strong limit},\ \mathrm{cf}(\delta) < \kappa\}$. For each $\delta \in A$, there is a function $q^\delta : P_\kappa\delta \to P_\kappa\delta$ so that $\mathrm{CF}_{\kappa\delta} \subset q^\delta_*(U_\delta) \cong U_\delta$ where U_δ is a normal measure on $P_\kappa\delta$. We shall sum up these U_δ's and $q^\delta_*(U_\delta)$'s with a suitable ultrafilter on λ.

LEMMA 1.2. *There exists a κ-complete ultrafilter on λ including $\{A\} \cup \mathrm{CF}_\lambda$. ($\mathrm{CF}_\lambda$ is the club filter on λ.)*

PROOF. Since λ is inaccessible, A is stationary. Hence we have a λ-complete filter $E = \{X \subset \lambda : A - X \text{ is not stationary}\}$. It is easily seen that $\{A\} \cup \mathrm{CF}_\lambda \subset E$. Then the strong compactness of κ gives us a κ-complete ultrafilter D extending E.

We use the above D. Define F_1 and F_2 by

$$X \in F_1 \quad \text{if } X \subset P_\kappa\lambda \text{ and } \{\delta \in A : X \cap P_\kappa\delta \in U_\delta\} \in D,$$
$$X \in F_2 \quad \text{if } X \subset P_\kappa\lambda \text{ and } \{\delta \in A : X \cap P_\kappa\delta \in q^\delta_*(U_\delta)\} \in D.$$

F_1 and F_2 are fine measures on $P_\kappa\lambda$. We want to show that they are isomorphic and contain $\mathrm{CF}_{\kappa\lambda}$. The next is an easy but key lemma.

LEMMA 1.3. *Assume that* $\mathrm{cf}(\eta) < \kappa$ *and* U *is a fine measure on* $P_\kappa\eta$. *Then* $\{x \in P_\kappa\eta : \sup(x) = \eta\} \in U$.

PROOF. Let $\{\eta_\alpha : \alpha < \mathrm{cf}(\eta)\}$ be a cofinal subset of η. Since U is fine, $\{x : \eta_\alpha \in x\} \in U$ for each $\alpha < \mathrm{cf}(\eta)$. Using the κ-completeness of U and the fact that $\mathrm{cf}(\eta) < \kappa$, we get $\{x : \eta_\alpha \in x \text{ for every } \alpha < \mathrm{cf}(\eta)\} \in U$. □

COROLLARY 1.4. *For every* $\delta \in A$, $\{x \in P_\kappa\delta : \sup(x) = \delta\} \in U_\delta$ *and* $\{x \in P_\kappa\delta : \sup(q^\delta(x)) = \delta\} \in U_\delta$.

PROOF. Since $q^\delta_*(U_\delta)$ is also a fine measure on $P_\kappa\delta$ and $\mathrm{cf}(\delta) < \kappa$, $\{x : \sup(x) = \delta\} \in q^\delta_*(U_\delta)$. This is equivalent to $\{x : \sup(q^\delta(x)) = \delta\} \in U_\delta$. □

For $x \in P_\kappa\lambda$, let $\delta_x =$ the least member of A such that $x \in P_\kappa\delta$. And $q : P_\kappa\lambda \to P_\kappa\lambda$ is defined by

$$q(x) = q^{\delta_x}(x).$$

By our construction,

LEMMA 1.5. *For every* $\delta \in A$, $\{x \in P_\kappa\delta : \delta_x = \delta\} \in U_\delta$; *hence* $\{x : q(x) = q^\delta(x)\} \in U_\delta$.

We can see that F_1 and F_2 are isomorphic.

LEMMA 1.6. q *is one-to-one on a set in* F_1.

PROOF. Let $B_\delta \in U_\delta$ be such that q^δ is one-to-one on B_δ. We have already known that $C_\delta = \{x \in B_\delta : q(x) = q^\delta(x),\ \sup(q^\delta(x)) = \sup(x) = \delta\} \in U_\delta$. Hence $C = \bigcup\{C_\delta : \delta \in A\}$ is a member of F_1.

Suppose that x, $y \in C$ and $q(x) = q(y)$. There is a $\delta \in A$ such that $\delta = \sup(x) = \sup(q(x)) = \sup(q(y)) = \sup(y)$. Since x and y are in the same C_δ and $q \restriction C_\delta = q^\delta \restriction C_\delta$ is one-to-one, we have $x = y$. Thus q is one-to-one on $C \in F_1$. □

LEMMA 1.7. $F_2 = q_*(F_1)$.

PROOF. Recall that $X \in F_2$ iff $\{\delta \in A : X \cap P_\kappa\delta \in q^\delta_*(U_\delta)\} \in D$, and that $X \cap P_\kappa\delta \in q^\delta_*(U_\delta)$ is equivalent to $\{x \in P_\kappa\delta : q^\delta(x) \in X \cap P_\kappa\delta\} \in U_\delta$. By 1.5, the last paraphrase is the same as $\{x \in P_\kappa\delta : q(x) \in X\} \in U_\delta$.

Let $Y = \{x \in P_\kappa\lambda : q(x) \in X\}$. We have shown that $X \in F_2$ is equivalent to $\{\delta \in A : Y \cap P_\kappa\delta \in U_\delta\} \in D$. The latter says that $Y \in F_1$ and $X \in q_*(F_1)$. Hence $X \in F_2$ iff $X \in q_*(F_1)$. □

What is left to show is that both F_1 and F_2 contain $\mathrm{CF}_{\kappa\lambda}$. Note that $\{\delta < \lambda : X \cap P_\kappa\delta \in \mathrm{CF}_{\kappa\delta}\} \in \mathrm{CF}_\lambda$ for every $X \in \mathrm{CF}_{\kappa\lambda}$.

LEMMA 1.8. $\mathrm{CF}_{\kappa\lambda} \subset F_1 \cap F_2$.

PROOF. Suppose that $X \in \mathrm{CF}_{\kappa\lambda}$. Then $X' = \{\delta < \lambda : X \cap P_\kappa\delta \in \mathrm{CF}_{\kappa\delta}\} \in \mathrm{CF}_\lambda \subset D$. Since U_δ and $q^\delta_*(U_\delta)$ contain $\mathrm{CF}_{\kappa\delta}$, $X \cap P_\kappa\delta$ belongs to both U_δ and $q^\delta_*(U_\delta)$ for all $\delta \in X'$. Hence $X \in F_1 \cap F_2$. □

Now we are done.

THEOREM 1.9. *If* λ *is a strongly inaccessible cardinal greater than* κ *a supercompact, there are two distinct isomorphic fine measures on* $P_\kappa\lambda$ *containing the club filter.*

The author does not know whether a normal measure on $P_\kappa\lambda$ is isomorphic to a fine measure containing $\mathrm{CF}_{\kappa\lambda}$ under the same assumption. It is also still open whether two fine measures can be isomorphic for a successor cardinal λ. The case that λ is not strong limit is also open.

2. $\mathrm{SCF}_{\kappa\lambda}$, **prestationary sets and the partition property.** For the subsets of regular uncountable cardinals, the situation is simple. That is, $S \subset \kappa$ is stationary iff for any regressive function f on S, there is an unbounded set $T \subset S$ on which f is constant. But this does not hold for the subsets of $P_\kappa\lambda$.

In this section, κ is a regular uncountable cardinal and $\lambda > \kappa$. We begin by Menas' invention again.

PROPOSITION 2.1 (MENAS [**8**]). *There is a nonstationary subset S of $P_\kappa\lambda$ such that every regressive function is constant on an unbounded subset of S.*

DEFINITION 2.2. We call such a set S "prestationary".

Menas characterized S 'stationary" as follows:

PROPOSITION 2.3 (MENAS [**8**]). *$S \subset P_\kappa\lambda$ is stationary iff any function $f: S \to \lambda \times \lambda$ so that $f(y) \in y \times y$ for all y in S, is constant on some unbounded $T \subset S$.*

In the spirit of Proposition 2.3, we can express stationarity using prestationarity.

PROPOSITION 2.4. *If $S \subset P_\kappa\lambda$ is prestationary and every regressive function is constant on a prestationary $T \subset S$, then S is stationary.*

PROOF. Let $f: S \to \lambda \times \lambda$, f_1, $f_2: S \to \lambda$ so that $f(y) \in y \times y$ for all $y \in S$ and $f(y) = (f_1(y), f_2(y))$. Since $f_1(y) \in y$ for all $y \in S$, there is a prestationary $T_1 \subset S$ on which f_1 is constant. Again by the fact that $f_2(y) \in y$ for every $y \in T_1$ that is prestationary, there is an unbounded $T_2 \subset T_1$ so that $f_2 \restriction T_2$ is constant. Then $f \restriction T_2$ is constant. □

The stationary subsets are the sets which have nonempty intersection with every closed unbounded set. Now we characterize the prestationary sets with $\mathrm{SCF}_{\kappa\lambda}$. First recall the theorem for $\mathrm{SCF}_{\kappa\lambda}$ in Carr [**3**].

LEMMA 2.5 (CARR). *$C \in \mathrm{SCF}_{\kappa\lambda}$ iff there is a sequence of sets in $P_\kappa\lambda$, $\langle x_\alpha | \alpha < \lambda\rangle$ so that $\Delta\langle \hat{x}_\alpha | \alpha < \lambda\rangle = \{y: x_\alpha \subset y \text{ for all } \alpha \in y\} \subset C$.*

PROPOSITION 2.6. *$S \subset P_\kappa\lambda$ is prestationary iff $S \cap C \neq 0$ for all $C \in \mathrm{SCF}_{\kappa\lambda}$.*

PROOF. Suppose that S is prestationary and $S \cap C = 0$ for some $C \in \mathrm{SCF}_{\kappa\lambda}$. By 2.5, there is a sequence $\langle x_\alpha | \alpha < \lambda\rangle$ so that $\Delta\langle \hat{x}_\alpha | \alpha < \lambda\rangle \subset C$. For every $x \in S$, there exists an $\alpha \in x$ such that $x_\alpha \not\subset x$. Since S is prestationary, there is an ordinal γ so that $\{x \in S: x_\gamma \not\subset x\}$ is unbounded. Contradiction.

For the converse, assume that $S \cap C \neq 0$ for all $C \in \mathrm{SCF}_{\kappa\lambda}$ and S is not prestationary. There is a regressive function f such that for every $\alpha < \lambda$ there is an $a_\alpha \in P_\kappa\lambda$ so that $\{x \in S: f(x) = \alpha\} \cap \hat{a}_\alpha = 0$. Let $C = \Delta\{\hat{a}_\alpha | \alpha < \lambda\}$; then $C \in \mathrm{SCF}_{\kappa\lambda}$. Pick an $x \in C \cap S$ and suppose that $f(x) = \alpha$. Since $\alpha \in x$ and $x \in C$, $a_\alpha \subset x$. Then $f(x) \neq \alpha$ by the definition of a_α. This is absurd. □

We connect the above fact to the partition property of fine measures.

COROLLARY 2.7. *If U is a fine measure with the partition property assigning measure one to the strongly club sets, then U is normal.*

This is really Proposition 11 in Menas [**9**], where he proved it for the club sets version. Menas' proof is applicable in our case as well.

3. Weakly normal filters on $P_\kappa\lambda$. For weakly normal filters on κ regular, see Kanamori [**7**]. We briefly review the basic facts.

PROPOSITION 3.1. *For any filter on κ, the following are equivalent.*

(i) *U is weakly normal.*

(ii) *Every filter extension of U is weakly normal.*

(iii) *If $\{X_\alpha : \alpha < \kappa\}$ are sets of positive measure such that $X_\beta \subset X_\alpha$ whenever $\alpha < \beta$, then $\Delta\{X_\alpha : \alpha < \kappa\} = \{\alpha < \kappa : \alpha \in X_\beta$ for all $\beta < \alpha\}$ has a positive measure.*

(iv) *U is a p-point filter extending CF_κ. (U is a p-point if every function $f : \kappa \to \kappa$ such that $\kappa - f^{-1}(\{\alpha\}) \in U$ for all $\alpha < \kappa$ is $< \kappa$ to one on some $X \in U$.)*

It is natural to ask whether the same thing happens to filters on $P_\kappa\lambda$. We easily get that (i)$\sim$(iii) are also equivalent for any filter on $P_\kappa\lambda$. (Note that $\Delta\{X_\alpha : \alpha < \lambda\} = \{x \in P_\kappa\lambda : x \in X_\alpha$ for all $\alpha \in X\}$.)

But for (iv), the author only knows the following.

PROPOSITION 3.2. (i) *Suppose that U is weakly normal. If f is a function with the domain $P_\kappa\lambda$ and $\{x : f(x) > \alpha\} \in U^+$ for all $\alpha < \lambda$, then there is a set X of positive measure so that $X \cap f^{-1}(\{\alpha\}) \subset P_\kappa\alpha$ for all $\alpha < \lambda$.*

(ii) *Suppose that U extends $\mathrm{SCF}_{\kappa\lambda}$ and for any $\alpha < \lambda$ there is an $X \in U^+$ such that $X \cap f^{-1}(\{\alpha\}) \subset P_\kappa\beta$ for some $\beta < \lambda$ whenever f satisfies $\{x : f(x) > \gamma\} \in U^+$ for all $\gamma < \lambda$. Then U is weakly normal.*

PROOF. (i) Let $X_\xi = \{x : f(x) > \xi\}$ for each $\xi < \lambda$. Then $X_\xi \in U^+$ and $X_\eta \subset X_\xi$ if $\xi < \eta$. Now $\Delta\{X_\xi : \xi < \lambda\} \in U^+$ by (iii). If $x \in \Delta\{X_\xi : \xi < \lambda\}$ and $f(x) = \alpha$, then $\xi < \alpha$ for all $\xi \in x$. Hence $x \subset \alpha$.

(ii) Suppose that f is a regressive function on $P_\kappa\lambda$. Since U extends $\mathrm{SCF}_{\kappa\lambda}$, every X of positive measure is prestationary. Hence there is an $\alpha < \lambda$ so that $X \cap f^{-1}(\{\alpha\})$ is unbounded. By our hypothesis, $\{x : f(x) < \gamma\} \in U$ for some $\gamma < \lambda$.

The question left is whether every weakly normal filter extends $\mathrm{CF}_{\kappa\lambda}$ or $\mathrm{SCF}_{\kappa\lambda}$. In [**1**], the fine measure investigated by Menas was revisited and shown to be nonnormal. We again observe it and get more information, which gives a negative answer to the question. The author wishes to express his gratitude to A. Blass whose advice led to a simplified proof. We concentrate on a filter defined below. We assume that κ is a regular limit cardinal.

Let $\langle U_\alpha | \alpha < \kappa\rangle$ be a sequence of fine filters on $P_\alpha\lambda$ and D be a κ-complete uniform filter on κ. Then a fine filter U is defined by $X \in U$ if $X \subset P_\kappa\lambda$ and $\{\alpha < \kappa : X \cap P_\alpha\lambda \in U_\alpha\} \in D$.

THEOREM 3.3 (INSPIRED BY BLASS). *U does not extend $\mathrm{SCF}_{\kappa\lambda}$ hence is nonnormal.*

PROOF. Let $C = \{x \in P_\kappa\lambda : x \cap \kappa$ is an ordinal$\}$. Then C is strongly closed unbounded. We shall show that $C \cap P_\alpha\lambda$ is not unbounded for all $\alpha < \kappa$. If $x \in C \cap P_\alpha\lambda$ and $\alpha^+ \in x$, then $\alpha^+ \subset x$. But this contradicts $|x| < \alpha$. Hence $\alpha^+ \notin x$ for all $x \in C \cap P_\alpha\lambda$ and $C \cap P_\alpha\lambda \notin U_\alpha$. Thus $C \notin U$. Note that $\alpha^+ < \kappa < \lambda$ since κ is a limit cardinal. □

For certain $A \subset \kappa$ we have a strongly club set which is not unbounded for any $\alpha \in A$. More precisely;

PROPOSITION 3.4. *Suppose that* $\lambda^{<\kappa} = \lambda$ *and* $A \subset \kappa$. *There is a* $C \in \mathrm{SCF}_{\kappa\lambda}$ *so that if* $\alpha \in A$ *and* $\sup(A \cap \alpha) \neq \alpha$, *then* $C \cap P_\alpha\lambda$ *is not unbounded.*

PROOF. Let $\{x_z : \xi < \lambda\}$ be an enumeration of $P_\kappa\lambda$ and $\alpha_z =$ the least member of $A > |x|$. Then, we pick a $y_z \supset x$ with $|y_z| \geq \alpha_\xi^+$. Finally, $C = \Delta\langle \hat{y}_z | \xi < \lambda\rangle$.

Suppose that $\alpha \in A$ and $\sup(A \cap \alpha) \neq \alpha$. Then $\alpha = \alpha_z$ for some x_z. Assume that there exists an $x \in C \cap P_\alpha\lambda$ with $\xi \in x$. By our definition of C, $x \supset y_z$. This implies $|x| \geq |y_z| \geq \alpha_\xi^+ > \alpha$ contradicting $x \in P_\alpha\lambda$. Hence $(C \cap P_\alpha\lambda) \cap \{\hat{\xi}\} = 0$. □

Now we turn to the weak normality of U under the assumption that U_α is weakly normal for all $\alpha < \kappa$, and improve Proposition 2.4 in [**1**] by a simple argument. In the next theorem, κ is not necessarily a limit cardinal in (i) and (iii).

THEOREM 3.5. (i) *If* $\mathrm{cf}(\lambda) > \kappa$, *then* U *is weakly normal.*

(ii) *If* $\mathrm{cf}(\lambda) = \kappa$, *then* U *is not weakly normal.*

(iii) *If* $\mathrm{cf}(\lambda) < \kappa$ *and* (a) *or* (b) *is satisfied, then* U *is weakly normal.*

(a) U *is an ultrafilter.*

(b) D *is* $\mathrm{cf}(\lambda)$*-descendingly complete. That is; if* $\langle X_\xi | \xi < \mathrm{cf}(\lambda)\rangle$ *is a sequence of positive measure such that* $X_\eta \subset X_\xi$ *whenever* $\xi < \eta$, *then* $\bigcap\{X_\xi : \xi\ \mathrm{cf}(\lambda)\} \neq 0$. (*Note that* D *is not required to be an ultrafilter.*)

PROOF. Suppose that $f(x) \in x$ for every $x \in P_\kappa\lambda$.

(i) For $\alpha < \kappa$, δ_α is an ordinal $< \lambda$ such that $\{x \in P_\alpha\lambda : f(x) < \delta_\alpha\} \in U_\alpha$. Since $\mathrm{cf}(\lambda) > \kappa$, $\delta = \sup(\{\delta_\alpha : \alpha < \kappa\}) < \lambda$. Obviously $\{x \in P_\kappa\lambda : f(x) < \delta\} \in U$.

(ii) Let $\{\lambda_\alpha : \alpha < \kappa\}$ be a cofinal subset of λ and $\lambda_\alpha < \lambda_\beta$ if $\alpha < \beta$. For each $\alpha < \kappa$, $\{x \in P_\alpha\lambda : \lambda_\alpha \in x \text{ and } \lambda_{|x|} < \lambda_\alpha\} \in U_\alpha$. Hence we have $\{x \in P_\kappa\lambda : x - \lambda_{|x|} \neq 0\} \in U$.

So, there is a function $g : P_\kappa\lambda \to \lambda$ such that $g(x) \in x$ and $g(x) > \lambda_{|x|}$ for almost all x (mod U). For any $\alpha < \kappa$, we know that $\{x \in P_\alpha\lambda : x \supset \alpha^+\} \in U$ and then $\{x : \lambda_{|x|} > \lambda_\alpha\} \in U$. Hence $\{x \in P_\kappa\lambda : g(x) > \lambda_\alpha\} \in U$ for every $\alpha < \kappa$. We are done because g is an unbounded regressive function.

(iii) Suppose that (a) holds. We already showed in Lemma 1.3 that every fine measure on $P_\kappa\lambda$ is weakly normal if $\mathrm{cf}(\lambda) < \kappa$. In fact,

Fact 3.6. A fine measure is weakly normal iff its first function maps x to $\sup(x)$. (We denote such a function by Sup.)

When (b) holds, let $\{\lambda_\alpha : \alpha < \delta\}$ be a cofinal subset of λ with $\delta = \mathrm{cf}(\lambda)$ so that $\lambda_\alpha < \lambda_\beta$ if $\alpha < \beta$. Suppose that $\{x \in P_\kappa\lambda : f(x) < \lambda_\alpha\} \notin U$ for all $\alpha < \delta$. Then $\{\xi < \kappa : \{x \in P_\xi\lambda : f(x) < \lambda_\alpha\} \in U_\xi\} \notin D$ for any $\alpha < \delta$. Hence

$$C_\alpha = \{\xi < \kappa : \{x \in P_\xi\lambda : f(x) < \lambda_\alpha\} \notin U_\xi\} \in D^+.$$

If $\alpha < \beta$, then $\{x \in P_\xi\lambda : f(x) < \lambda_\beta\} \notin U_\xi$ implies $\{x \in P_\xi\lambda : f(x) < \lambda_\alpha\} \notin U_\xi$ since $\lambda_\alpha < \lambda_\beta$. So, $C_\beta \subset C_\alpha$. Then $C = \bigcap\{C_\alpha : \alpha < \delta\} \neq 0$.

Pick a $\xi \in C$. $\{x \in P_\xi\lambda : f(x) < \lambda_\alpha\} \notin U_\xi$ for any $\alpha < \delta$. This contradicts the hypothesis that U_ξ is weakly normal. □

Note that a filter F on $P_\kappa\lambda$ is weakly normal if it is $\mathrm{cf}(\lambda)$-descendingly complete. Combining Theorems 3.3 and 3.5, we have;

COROLLARY 3.7. *There is a weakly normal filter which does not extend* $\mathrm{SCF}_{\kappa\lambda}$.

Jech [**5**] and Carr [**3**] showed that $\mathrm{CF}_{\kappa\lambda}$ is the minimal normal filter. Is there a nice analogue for weakly normal filter? Or, what is the consistency of weakly normal filters? (Note here we assume that any filter is fine and κ-complete.)

4. Weakly normal fine measures and the RK-**ordering.** In this section, κ is a fixed strongly compact cardinal. We observe the weak normality in view of the RK-ordering. First we review the fact established by Menas in [**8**].

THEOREM 4.1 (MENAS). (i) *If* $\mathrm{cf}(\lambda) < \kappa$ *or* λ *is regular, then every normal measure on* $P_\kappa\lambda$ *is minimal.*

(ii) *If* λ *is regular and the first function of* U *is one-to-one on a set of measure one, then* U *is minimal.*

We hope that every weakly normal measure is minimal as in the theory of uniform ultrafilters on a regular cardinal. In fact any minimal fine measure is isomorphic to a weakly normal measure.

PROPOSITION 4.2. *Every fine measure has a weakly normal measure below it.*

PROOF. Let U be a fine measure and g its first function. Define $f: P_\kappa\lambda \to P_\kappa\lambda$ by $f(x) = x \cap g(x)$.

By an easy observation, $\{x: \alpha \in f(x)\} \in U$ for all $\alpha < \lambda$ and $f_*(U)$ is a fine measure.

Suppose that $\{x: f(x) \in x\} \in f_*(U)$. It means that $\{x: h \circ f(x) \in x \cap g(x)\} \in U$. Since g is the first function of U, we have $\{x: h \circ f(x) < \gamma\} \in U$ for some $\gamma < \lambda$. Hence $\{x: h(x) < \gamma\} \in f_*(U)$. □

The next fact appeared already in [**8**] implicitly.

PROPOSITION 4.3. *Let* λ *be regular and* U *a fine measure on* $P_\kappa\lambda$. U *is minimal iff its first function is one-to-one on a set* $X \in U$.

PROOF. Let $\{A_\lambda(\alpha): \alpha < \lambda\}$ be a partition of $\{\alpha < \lambda: \mathrm{cf}(\alpha) = \omega\}$ into disjointed stationary subsets. Let f be the first function and define q by $q(x) = \{\alpha < f(x): A_\lambda(\alpha) \cap f(x)$ is stationary in $f(x)\}$. Then $q_*(U)$ is a minimal fine measure (Theorem 2.14 in [**8**]).

Suppose that U is minimal. $q \restriction X$ is one-to-one for some $X \in U$. But $q(x) = q(y)$ if $f(x) = f(y)$. Hence $f \restriction X$ is one-to-one. □

COROLLARY 4.4. *A weakly normal measure on* $P_\kappa\lambda$ *with* λ *regular is minimal iff* Sup *is one-to-one on a set of measure one.*

A filter F on a regular cardinal ρ is called a q-point if every $< \rho$ to one function from ρ to ρ is one-to-one on a set $X \in F$. It is known that any filter extending CF_ρ is a q-point. $\mathrm{SCF}_{\kappa\lambda}$ also plays a role on the minimality of weakly normal measures.

PROPOSITION 4.5. *Let* λ *be regular. If* U *is a minimal fine measure on* $P_\kappa\lambda$ *that is not weakly normal, then* $\mathrm{SCF}_{\kappa\lambda} \not\subset U$.

PROOF. Let f be the first function. By our assumption, there is a set $X \in U$ so that $f \restriction X$ is one-to-one and $f(x) < \sup(x)$ for all $x \in X$.

Suppose that $\mathrm{SCF}_{\kappa\lambda} \subset U$. Then X is prestationary. For $x \in X$, set $g(x) =$ the least member of x greater than $f(x)$. There is an unbounded set $Y \subset X$ such that $g''Y = \{\gamma\}$ for some $\gamma < \lambda$. Thus, $f''Y \subset \gamma$ and $|Y| = \lambda^{<\kappa} > \gamma$, which contradicts the fact that $f \restriction Y$ is one-to-one. □

COROLLARY 4.6. *Let λ be regular. If U is normal and $f_*(U) \supset \mathrm{SCF}_{\kappa\lambda}$, then $f_*(U)$ is weakly normal and $\{x\colon \sup(f(x)) = \sup(x)\} \in U$.*

COROLLARY 4.7. *For any regular $\lambda > \kappa$, there is a nonminimal fine measure extending $\mathrm{CF}_{\kappa\lambda}$.*

PROOF. Let $A = \{\alpha < \lambda\colon \mathrm{cf}(\alpha) < \kappa\}$ which is stationary in λ. We repeat the construction in §1.

There is a κ-complete ultrafilter on λ, $D \supset \mathrm{CF}_\lambda \bigcup\{A\}$. For each $\alpha \in A$, fix a fine filter U_α on $P_\kappa\alpha$ extending $\mathrm{CF}_{\kappa\alpha}$, and define U by

$$X \in U \quad \text{iff } \{\alpha < \lambda\colon X \cap P_\kappa\alpha \in U_\alpha\} \in D.$$

Then U is a fine measure extending $\mathrm{CF}_{\kappa\lambda}$.

We shall see that U is not weakly normal, hence nonminimal by Proposition 4.5.

Since $A \in D$, D is not normal. Thus there is a function g so that $[g]_D = \lambda$ and $\{\alpha < \lambda\colon g(\alpha) < \alpha\} \in D$.

For $x \in P_\kappa\lambda$, let $\alpha_x =$ the least α such that $x \in P_\kappa\alpha$ and $f(x) = g(\alpha_x)$. For every $\alpha \in A$, $\{x \in P_\kappa\alpha\colon f(x) < \sup(x)\} \in U_\alpha$ since $\{x\colon \alpha_x = \alpha = \sup(x)\} \in U_\alpha$. Let $h(x) =$ the least member of x greater than $f(x)$. h is a regressive function on a set in U.

Pick a $\gamma < \lambda$. Then $B = \{\alpha \in A\colon \gamma < g(\alpha)\} \in D$. For all $\alpha \in B$, $\{x \in P_\kappa\alpha\colon f(x) = g(\alpha_x) = g(\alpha)\} \in U_\alpha$ and $\{x \in P_\kappa\alpha\colon f(x) > \gamma\} \in U_\alpha$. Hence $\{x \in P_\kappa\lambda\colon \gamma < f(x)\} \in U$. It shows that Sup is not the least function. □

On the other hand, we have a minimal fine measure which is weakly normal and does not extend $\mathrm{SCF}_{\kappa\lambda}$. We recall the fine measure in §3. Suppose that λ is regular and $\langle U_\alpha | \alpha < \kappa\rangle$ is a sequence of normal measures on $P_\alpha\lambda$ and D is a normal measure on κ. Define U by

$$X \in U \quad \text{iff } \{\alpha < \kappa\colon X \cap P_\alpha\lambda \in U_\alpha\} \in D.$$

Following the argument of 3.1, 3, 4 in [**10**], we get

LEMMA 4.8. (i) *$\{x$: the order type of x is regular$\} \in U$.*

(ii) *Let G be a ω-Jonsson function over λ. (G is ω-Jonsson over y if $G\colon {}^\omega y \to y$ and $G''z = y$ whenever $z \subset y$ and $|z| = |y|$.) Then we have $\{x\colon G \restriction {}^\omega x$ is ω-Jonsson over $x\} \in U$.*

(iii) *There is an $X \in U$ so that* Sup $\restriction X$ *is one-to-one.*

Note that normality of U_α's is necessary in the above. Using the results proved in §3, we can show

THEOREM 4.9. *For every regular $\lambda > \kappa$, there is a weakly normal minimal fine measure which does not extend* $\mathrm{SCF}_{\kappa\lambda}$.

PROOF. It is clear that every normal measure is weakly normal. Hence our U is weakly normal by Theorem 3.5(i). Theorem 3.3 asserts that U does not extend $\mathrm{SCF}_{\kappa\lambda}$. At last U is minimal by Fact 3.6, Theorem 4.1(ii), and Lemma 4.8(iii). □

It is not known whether U can be isomorphic to some fine measure extending $\mathrm{SCF}_{\kappa\lambda}$. We also do not know whether nonminimal weakly normal measures exist.

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