ON THE MONODROMY GROUP
OF EVERYWHERE TANGENT LINES
TO THE OCTIC SURFACE IN $\mathbb{P}^3$

HARRY D'SOUZA

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ABSTRACT. Let $S_0$ be an octic surface in $\mathbb{P}^3$, $G = G(1,3) = \text{Grassmannian}$
of lines in $\mathbb{P}^3$, and $J = \{(x, l) | x \in l \cap S_0\} \subset S_0 \times G$. Then $\dim J = 5$. Let
$L = \{l | l \text{ is everywhere tangent to } S_0\} \subset G$. Let $\pi_2 : S_0 \times G \rightarrow G$ be the
projection onto the second factor. We denote its restriction to $J$ also by $\pi_2$.
Then the locus of everywhere tangent lines is $\pi_2(L)$. In this article we show
that the monodromy group of these lines is the full symmetric group.

Introduction. In this article we show that a general octic surface in $\mathbb{P}^3$, has
a finite number of everywhere tangent lines and using the Harris technique in [H],
we study the monodromy group of these lines and prove that this group is the full
symmetric group.

Notation and background material.

(0.1) Let $S_0$ be an octic surface in $\mathbb{P}^3$. First we note that if $S_0$ is general then
it cannot contain lines. Let $G = G(1,3) = \text{Grassmannian}$ of lines in $\mathbb{P}^3$. Let
$J = \{(x, l) | x \in l \cap S_0\} \subset S_0 \times G$. We also note that $\dim J = 2 + 4 - 1 = 5$.

(0.2) DEFINITION. A line $l$ in $\mathbb{P}^3$ is said to be everywhere tangent to an octic
surface $S$ in $\mathbb{P}^3$, if the multiplicity at each of the points of intersection of $S$ and $l$
in $\mathbb{P}^3$ is at least two.

(0.3) Let $L = \{(x, l) | l \text{ is everywhere tangent to } S_0\} \subset J$; also let $\pi_2 : S_0 \times G \rightarrow G$
be the projection onto the second factor. We denote its restriction to $J$ also by $\pi_2$,
then the locus of everywhere tangent lines to $S_0$ is $\pi_2(L)$. From now on we denote
$\pi_2(L)$ by $F$.

(0.4) Let $X$ be a smooth complex projective variety of dimension $n$.

Let $\text{Hilb}^r_X$ denote the curvilinear Hilbert scheme of $r$ points of $X$. Then $\text{Hilb}^r_X$ is
smooth of dimension $nr$ (see [LB] for details).

If $Y$ is a smooth hypersurface in $X$, then there is a natural inclusion $\text{Hilb}^r_X Y \rightarrow$
$\text{Hilb}^r_X X$ of dimension $(n - 1)r$.

Let $\text{Hilb}^r_{S/P^N}$ denote the relative (curvilinear) Hilbert scheme of the universal
family of octic surfaces parametrized by $P^N$; we note that $N = 164$.

The linear system of all octics in $\mathbb{P}^3$ is denoted by $W_8$, and is of dimension
$N = 164$. If $S$ denotes a general octic surface in $\mathbb{P}^3$ (see (0.6)), then the relative
(curvilinear) Hilbert scheme above is of dimension $2r + N$.

(0.5) Let $\mathbf{a} = (a_1, \ldots, a_k)$ where $a_i \leq a_j$ where $i < j$ and $a_i \geq 2$, and $r = \sum a_i = 8$. We note that $\mathbf{a} = (2, 2, 2, 2); (2, 2, 4); (2, 3, 3); (2, 6); (3, 5); (4, 4); (8)$, are the only
possibilities.
In what follows, the notation is borrowed from [Co], where one can check for details.

Let \( a = (b_1, \ldots, b_1, b_2, \ldots, b_p, \ldots, b_p) \) where \( b_i < b_j \) for \( i < j \), and \( k = \sum q_i \) where \( q_i \) denotes the number of \( b_i \), for \( 1 \leq i \leq p \). Consider \( j_a : \text{Hilb}^{q_1} P^1 \times \cdots \times \text{Hilb}^{q_p} P^1 \to \text{Hilb}^r P^1 \cong P^r \) given by

\[
(I_1, \ldots, I_p) \mapsto I_1^{b_1}, \ldots, I_p^{b_p}
\]

where \( I_j \) is the ideal of the zero-dimensional length \( q_j \) subscheme of \( P^1 \). We note that \( j_a \) is an embedding. Using \( j_a \) we can construct a flat family over \( G \), denoted by \( D_a \), whose fibres are all isomorphic to the image of \( j_a \), and \( D_a \) embeds naturally in the curvilinear Hilbert scheme \( \text{Hilb}_c^r P^3 \), and \( \dim D_a = k + 4 \).

(0.6) DEFINITION. Consider \( P_a = D_a \cap \text{Hilb}_c^r S \), where \( S \) is an octic surface in \( P^3 \). \( S \) is said to be general if \( P_a \) is empty or \( P_a \) has the expected dimension \((k + 4) + 2r - 3r = k - 4 \) for all possible partitions of \( 8 = \deg S \).

General results.

(1.0) PROPOSITION. There exists a nonempty open subset \( U \) of \( W_8 (\cong P^N) N = 164 \), (see (0.4)) such that any member of \( U \) is general in the sense of (0.6).

PROOF. Consider the following commutative diagram

\[
\begin{array}{ccc}
X & \to & \text{Hilb}_c^r S/P^N \\
\downarrow & & \downarrow \\
D_a & \to & \text{Hilb}_c^r P^3 \\
\downarrow & & \\
G
\end{array}
\]

where \( X = D_a \times_{\text{Hilb}_c^r P^3} \text{Hilb}_c^r S/P^N \), is the fiber product. The fiber of \( X \) over \( l \) (\( \in G \)) consists of all type-\( a \) subschemes of \( l \) which also lies on some octic surface \( S \). Hence the dimension of the fiber is \((N - r) + k = 164 - 8 + k = 156 + k \). Also both \( l \) and \( S \) lie in some \( P^3 \). Hence \( \dim X = 160 + k \). Hence the general fiber over \( P^N \) is of dimension \( 160 + k - N = k - 4 \). But the fiber of a general \( x \in P^N \) is just \( P_a \) (see (0.6)). Hence for an open subset \( U \) of \( P^N \), \( P_a \) as desired. \( \square \)

(1.1) COROLLARY. If \( S_0 \) is a general octic in \( P^3 \) then \( F \) is a finite set. Moreover only the \((2,2,2,2)\) type (see (0.5)) can possibly occur.

PROOF. That \( F \) is finite is immediate from (1.0). Since \( S_0 \) is of degree 8, we have \( l.S_0 = 8 \) in \( P^3 \). Let \( a = (a_1, \ldots, a_k) \) where \( a_i \geq 2 \) and \( a_i \leq a_j \) for \( i < j \). Let \( r = \sum a_i = 8 \), where \( a_i \) denotes the multiplicity of \( l \) at the point of intersection with \( S_0 \), then it is easy to see that the set of \( l \) with \( a = (2,2,4); (2,3,3); (2,6); (3,5); (4,4) \) or \( 8 \) has \( k < 4 \), (see (0.5)) in each case and so for the open set \( U \) in (1.0) \( P_a \) is empty. In the case (2,2,2,2); \( k = 4 \), and so this can occur. \( \square \)

Main results. First we show that \( F \) is not empty, by showing the existence of a general octic surface with an everywhere tangent line. We need the following lemmas and this remark.

(2.0) REMARK. In the following we let \( u_8 \) = the linear system of all octics in \( P^2 \), and \( W_8 \) = the linear system of all octic surfaces in \( P^3 \).
LEMMA. Given a line \( l \) in \( \mathbb{P}^2 \) and 4 distinct points on it, there exists an octic curve, \( C \) in \( \mathbb{P}^2 \) such that \( m_p(l.C) = 2 \), i.e. the line intersects each of these points with multiplicity 2.

PROOF. If \( G = G(1,2) \) = Grassmannian of all lines in \( \mathbb{P}^2 \), then \( \dim G = 2 \) and \( \dim \omega_8 = 44 \). Let \( p_1, p_2, p_3 \) and \( p_4 \) denote the 4 specified points on \( l \).

Let \( w' \) be the space of octics having the tangent line \( l \) at these 4 points \( p_i \), then \( \dim w' = \dim w - [2 for each \( p_i \)] = 44 - 8 = 36 \). Also the space of all marked lines in \( \mathbb{P}^2 \) passing through 4 marked points \( p_i \) is of dimension, \( \dim \mathbb{P}^2 + [1 for each \( p_i \)] = 2 + 4 = 6 \).

Let \( w_0 \) be the space of octics with everywhere tangent lines in \( \mathbb{P}^2 \); then \( \dim w_0 = 36 + 6 = 42 \). □

LEMMA. Given a line \( l \) in \( \mathbb{P}^2 \) and 3 distinct points say \( p', p'' \) and \( p''' \) on it, there exists an octic curve, \( C \) in \( \mathbb{P}^2 \) such that \( m_{p'}(l.C) = 2 = m_{p''}(l.C) \) and \( m_{p'''}(l.C) = 4 \).

PROOF. Let \( w' \) be as in (2.1). Then as in (2.1) we see that \( \dim w' = \dim w - [2 for each of \( p' \) and \( p'' \) and 4 for \( p''' \)] = 44 - 8 = 36 \) and the space of all marked lines in \( \mathbb{P}^2 \) passing through 3 marked points \( p', p'' \) and \( p''' \) is of dimension, \( \dim \mathbb{P}^2 + [1 for each of \( p', p'' \) and \( p''' \)] = 2 + 3 = 5 \). Thus the dimension of the space of such octics is \( 36 + 5 = 41 \). □

(2.2.1) Let \( w_1 \) be the space of octics satisfying (2.2) then by (2.2), \( \dim w_1 = 41 \).

LEMMA. For a general octic surface in \( \mathbb{P}^3 \), there are only finitely many \( l \) with \( a = (2,2,2,2) \).

PROOF. Consider \( (\mathbb{P}^3)^* \), and let \( \zeta \) be the tautological bundle on \( (\mathbb{P}^3)^* \). This bundle is nontrivial. Choose a Zariski open subset, say \( A \), of \( (\mathbb{P}^3)^* \) over which this bundle is trivial.

For \( H \in A \), let \( w_S = \{ C_H | C_H = H \cap S, S \) a fixed general octic in \( \mathbb{P}^3 \} \subset \omega_S \) (see (2.0)), then via the map \( \Psi : A \rightarrow w_S \) given by \( H \mapsto H \cap S \) we see that \( \dim w_S = 3 \).

Hence if \( a = (2,2,2,2) \), by (2.1), \( \dim (w_0 \cap w_S) \geq \dim w_0 + \dim w_S - \dim \omega_8 = 42 + 3 - 44 = 1 \). Hence \( F \) is (1.0) is a nonempty finite set. □

LEMMA. Using the notations of (1.1) there exists an octic surface in \( \mathbb{P}^3 \), and a tangent line \( l \) with \( a = (0,2,2,4) \).

PROOF. We note that by (1.1), \( S_1 \) cannot be general. Let \( a = (0,2,2,4) \) and \( w_1 \) as in (2.2.1); suppose \( C_1 \in w_1 \). Let \( S_1 \) be an octic surface in \( \mathbb{P}^3 \) containing \( C_1 \), such a surface \( S_1 \) must exist by (2.2). □

(2.5) Let \( W_8 \) be as in (2.0). We note that \( \dim W_8 = 164 \). Let \( S \) be a general octic in \( \mathbb{P}^3 \); let \( m_p(S.l) \) denote the intersection multiplicity of \( S \) and \( l \), in \( \mathbb{P}^3 \), and let \( I_8 = \{(S; p, l)|l \) is everywhere tangent to \( S \), and \( p \in l \cap S\} \subset W_8 \times \mathbb{P}^3 \times G \). Let \( \pi : I_8 \rightarrow W_8 \) and \( \eta : I_8 \rightarrow \mathbb{P}^3 \times G \) be the projections onto the first and the last two factors respectively.

(2.6). REMARKS. (i) By (1.1), (2.1) and (2.2), \( \pi \) is generically finite.

(iii) Since the fibers of \( \eta \) are linear spaces of dimension 157, \( I_8 \) is irreducible.

(2.7) LEMMA. The monodromy group \( M, \) of \( \pi \) is twice transitive.

PROOF. First we show that \( M \) is transitive. Let \( U \) be a Zariski open subset of \( W_8 \) over which \( \pi \) is unramified, then \( V = \pi^{-1}(U) \) is irreducible. Let \( S \in U, \)
and let \((S; p', l')\) and \((S; p'', l'')\) \(\in \pi^{-1}(S)\) with \(m_{p'}(S, l') = 2\) and \(m_{p''}(S, l'') = 2\), respectively. Draw an arc \(\gamma\) on \(V\) with \(\gamma(0) = (S; p', l')\) and \(\gamma(1) = (S; p'', l'')\); then the monodromy action associated to the arc \(\gamma = \pi \circ \gamma\) will carry \((S; p', l')\) to \((S; p'', l'')\). Thus \(M\) is transitive.

Now let \(S_0\) be a general octic with \(p_0 \in S_0\), and \((p_0, l_0)\) such that \(m_p(S_0) = 2\) and \((S_0; p_0, l_0) \in I_8\), i.e. \(l_0\) is also everywhere else tangent to \(S_0\). Let \(W_0 = \pi^{-1}(p_0, l_0)\) be \(2\leq W_0\); we note that \(S_0 \in W_0\) and let \(I = \{(S; p, l) | m_p(S, l) \geq 2, p \neq p_0, l \neq l_0\}\ \subset I_8\), and \(\eta\) maps \(I\) onto a Zariski open subset \(\{(p, l) | p \neq p_0, l \neq l_0\}\) of \(P^3 \times G\), and as in (2.6), \(I\) is irreducible. Hence every Zariski open subset of \(I\) is connected. Suppose \((S_0; p', l'), (S_0; p'', l'')\) are two other points lying in \(\pi^{-1}(S_0)\), then we can find an arc \(\gamma\) in \(I\) such that \(\gamma(0) = (S_0; p', l')\) and \(\gamma(1) = (S_0; p'', l'')\). Note that \(\pi^{-1}(S_0)\) has all points except \((S_0; p_0, l_0)\) in \(I\); then the monodromy action associated to the arc \(\gamma = \pi \circ \gamma\) carries \((S_0; p', l')\) to \((S_0; p'', l'')\) and leaves \((S_0; p_0, l_0)\) fixed. Hence the stabilizer of \((S_0; p_0, l_0)\) in \(M\) acts transitively on the remaining points. Hence \(M\) is twice transitive. \(\Box\)

(2.8) LEMMA. Let \(M\) be as in (2.7), then \(M\) contains a simple transposition.

PROOF. By (2.4) we know that there exists an octic surface \(S_1\) in \(W_8\), with a line \(l\), such that \(m_{p'}(S_1, l) = 2 = m_{p''}(S_1, l)\) and \(m_{p'''}(S_1, l) = 4\). Let \(U\) be a Zariski open subset of \(W_8\), over which \(\pi\) is unramified. Let \(V = \pi^{-1}(U)\), and let \(\Delta\) be an open neighbourhood of \(S_1\) in \(W_8\), so chosen, so that the points \((S_1; p', l), (S_1; p'', l)\) and \((S_1; p''', l)\) have disjoint neighbourhoods \(\Delta', \Delta''\) and \(\Delta'''\) respectively, each mapping onto \(\Delta\), via \(\pi\). Since \(\Delta''\) is irreducible, \(\Delta''' \cap V\) is connected. Since \(I_8\) is irreducible, it is locally irreducible. Thinking of \((S_1; p''', l)\) as two infinitely near points \(q\) and \(r\) each with multiplicity two i.e. \(m_q(S_1, l) = 2 = m_r(S_1, l)\), we can find \(\gamma\) in \(\Delta''' \cap V\) such that \(\gamma(0) = (S_1; q, l)\) and \(\gamma(1) = (S_1; r, l)\); hence \(\gamma = \pi \circ \gamma\) will fix \((S_1; p', l)\) and \((S_1; p'', l)\) and \(\gamma\) interchanges \(q\) and \(r\). Hence \(M\) contains a simple transposition. \(\Box\)

(2.9) THEOREM. The monodromy group \(M\), of everywhere tangent lines to a general octic surface in \(P^3\), is the full symmetric group.

PROOF. By (2.7) and (2.8), for a general octic surface, the monodromy group \(M\), is twice transitive and contains a simple transposition; hence it contains all simple transpositions. So \(M\) is necessarily the entire symmetric group. \(\Box\)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, FLINT, MICHIGAN 48502-2186