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ON THE YOUNG-FENCHEL TRANSFORM FOR CONVEX FUNCTIONS

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ABSTRACT. Let $\Gamma(X)$ be the proper lower semicontinuous convex functions on a reflexive Banach space X. We exhibit a simple Vietoris-type topology on $\Gamma(X)$, compatible with Mosco convergence of sequences of functions, with respect to which the Young-Fenchel transform (conjugate operator) from $\Gamma(X)$ to $\Gamma(X^*)$ is a homeomorphism. Our entirely geometric proof of the bicontinuity of the transform halves the length of Mosco's proof of sequential bicontinuity, and produces a stronger result for nonseparable spaces.

1. Introduction. Let $\Gamma(X)$ denote the proper lower semicontinuous convex functions on a normed linear space X. Without question, for reflexive X, the fundamental notion of convergence for sequences in $\Gamma(X)$ is Mosco convergence, introduced by U. Mosco in [**12**].

DEFINITION. Let X be a normed linear space. A sequence of lower semicontinuous proper convex functions $\langle f_n \rangle$ on X is declared *Mosco convergent* to $f \in \Gamma(X)$ provided at each x in X.

(i) there exists a sequence $\langle x_n \rangle$ convergent strongly to x for which $\lim f_n(x_n) = f(x)$, and

(ii) whenever $\langle x_n \rangle$ converges weakly to x, then $\liminf f_n(x_n) \geq f(x)$.

The importance of Mosco convergence in the reflexive setting stems from its stability with respect to duality. With this notion of convergence, the Young-Fenchel transform, i.e., the conjugate operator, is "continuous": if $\langle f_n \rangle$ is Mosco convergent to f, then $\langle f_n^* \rangle$ is Mosco convergent to f^* ([**13**, **7**], and in finite dimensions, [**16** and **15**]).

Mosco convergence of functions is really a special case of a notion of set convergence, identifying elements of $\Gamma(X)$ with their epigraphs, as introduced by Mosco [**12**, Lemma 1.10]. Specifically, a sequence $\langle C_n \rangle$ of closed convex sets in a reflexive space X is declared *Mosco convergent* to a closed convex set C provided (i) at each x in C there exists a sequence $\langle x_n \rangle$ convergent strongly to x such that for each n, $x_n \in C_n$, and (ii) whenever $\langle n(i) \rangle$ is an increasing sequence of positive integers and $x_{n(i)} \in C_{n(i)}$ for each i, then the weak convergence of $\langle x_{n(i)} \rangle$ to $x \in X$ implies $x \in C$. In finite dimensions this reduces to Kuratowski convergence of sets, familiar to any point-set topologist [**9**]. For further information on Mosco convergence and its applications, one may consult [**1** or **14**].

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In [**2**], this author introduced a simple geometrically defined topology τ_M on the nonempty closed convex subsets $\mathscr{C}(X)$ of a normed linear space X compatible with Mosco convergence of sequences in $\mathscr{C}(X)$, without reflexivity [**2**, Theorem 3.1]. (The reader may also consult [**1**], where a completely different analytical approach to topologizing Mosco convergence of sequences may be found.) When X is reflexive, the topology is Hausdorff and completely regular [**2**, Theorem 3.4]. To describe this topology, we need some notation. For each $E \subset X$, we introduce these subsets E^- and E^+ of $\mathscr{C}(X)$

$$E^- = \{C \in \mathscr{C}(X) \colon C \cap E \neq \varnothing\} \quad \text{and} \quad E^+ = \{C \in \mathscr{C}(X) \colon C \subset E\}.$$

DEFINITION. Let $\mathscr{C}(X)$ be the nonempty closed convex subsets of a normed linear space X. The *Mosco topology* τ_M on $\mathscr{C}(X)$ is the topology generated by all sets of the form V^- where V is open in X and $(K^C)^+$ where K is a weakly compact subset of X.

Intuitively, V^- consists of those convex sets that "hit" V, whereas $(K^C)^+$ consists of those convex that "miss" the weakly compact set K. There are several other such "hit-and-miss" topologies in the literature [**2**, **3**, **4**, **8**]. Of greatest interest to topologists is the much stronger *finite* or *Vietoris topology* [**10**], which, in terms of applications, is highly pathological. When X is reflexive, it can be shown that the topology τ_M is first countable if and only if X is separable. But much more comes with separability: $\langle \mathscr{C}(X), \tau_M \rangle$ is a *Polish space*, i.e., the hyperspace is completely metrizable and separable [**2**, Theorem 4.3].

Since lower semicontinuous functions have closed epigraphs, we may view $\Gamma(X)$ as a topological subspace of $\langle \mathscr{C}(X \times R), \tau_M \rangle$. As such, a subbase for the Mosco topology on $\Gamma(X)$ consists of all sets of the form $V^- \cap \Gamma(X)$ where V is open in $X \times R$ and $(K^C)^+ \cap \Gamma(X)$ where K is a weakly compact subset of $X \times R$. As a special case of the compatability of Mosco convergence of sequences of convex sets with the topology τ_M, Mosco convergence of sequences in $\Gamma(X)$ is compatible with τ_M on $\mathscr{C}(X \times R)$, identifying functions with their epigraphs.

Again suppose that X is reflexive. Since the space $\langle \Gamma(X), \tau_M \rangle$ is first countable if and only if X is separable, Mosco's "continuity" theorem for the Young-Fenchel transform is really only a sequential continuity theorem unless X is separable, for only then do sequences determine the topology. In this note, we show that the transform is actually continuous from $\langle \Gamma(X), \tau_M \rangle$ to $\langle \Gamma(X^*), \tau_M \rangle$. Our proof is entirely geometric, and there is no need to consider limits inferior and limits superior of nets of sets, either explicitly or implicitly, via Lemma 1.10 of [**12**]. Even more attractive is the shortness of our proof, as compared with Mosco's proof. Finally, we follow Mosco's path to establish a true continuity theorem for the polar operation.

2. Preliminaries and additional notation. In the sequel, X will be a normed linear space with continuous dual X^*, often reflexive. The origin and closed unit ball of X (resp. X^*) will be represented by θ and B (resp. θ^* and B^*). If $C \in \mathscr{C}(X)$ its *polar* C° is the following subset of X^*:

$$C^\circ = \{y \in X^* \colon \text{ for each } x \in C, \ \langle y, x \rangle \leq 1\}.$$

We denote the projection map $(x, \alpha) \to x$ from $X \times R$ to X by π.

The *epigraph* of a convex function $f\colon X \to [-\infty,\infty]$ is the following convex subset of $X \times R$:

$$\operatorname{epi} f = \{(x,\alpha)\colon x \in X,\ \alpha \in R,\ \text{and } \alpha \geq f(x)\}.$$

Such a set is a closed subset of $X \times R$ if and only if f is lower semicontinuous [**5**, p. 103]. Dually, the *hypograph* of f is

$$\operatorname{hypo} f = \{(x,\alpha)\colon x \in X,\ \alpha \in R,\ \text{and } \alpha \leq f(x)\}.$$

A convex function $f\colon X \to [-\infty,\infty]$ is called *proper* provided its epigraph is nonempty and contains no vertical lines. As mentioned earlier, $\Gamma(X)$ will denote the proper lower semicontinuous convex functions on X. $\operatorname{Aff}(X)$ will denote the continuous real affine functions on X. If $C \in \mathscr{C}(X)$, then its *indicator function* $I(\cdot,C)$, defined by

$$I(x,C) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C \end{cases}$$

is in $\Gamma(X)$, whereas the *support function* $s(\cdot,C)$ of C, defined by

$$s(y,C) = \sup\{\langle y,x\rangle\colon x \in C\} \qquad (y \in X^*)$$

is in $\Gamma(X^*)$. If $f \in \Gamma(X)$ and $\alpha \in R$, we denote its *sublevel set at height* α, i.e., $\{x \in X\colon f(x) \leq \alpha\}$, by $\operatorname{sub}(f;\alpha)$.

For each $f \in \Gamma(X)$, we define its *conjugate* $f^*\colon X^* \to [-\infty,\infty]$ by the formula

$$f^*(y) = \sup\{\langle y,x\rangle - f(x)\colon x \in X\}.$$

It is well known that $f^* \in \Gamma(X^*)$, and $f^{**} = f$, provided X is reflexive [**6**, §14]. We will repeatedly use this fundamental fact: $(y,\alpha) \in \operatorname{epi} f^*$ if and only if f majorizes the continuous affine functional $x \to \langle y,x\rangle - \alpha$. The map $f \to f^*$ is called the *Young-Fenchel transform.*

We will need a slightly different description of the Mosco topology on $\Gamma(X)$, as provided next.

LEMMA 2.1. *Let X be a normed linear space. The Mosco topology τ_M on $\Gamma(X)$ is generated by all sets of the form $\Gamma(X) \cap (W \times (-\infty,\alpha))^-$ where W is open in X and $\Gamma(X) \cap (K^C)^+$ where K is a weakly compact subset of $X \times R$.*

PROOF. Suppose $f \in \Gamma(X) \cap V^-$ where V is open in $X \times R$. Choose $(x,\beta) \in \operatorname{hypo} f \cap V$. Since V is open, there exists $\alpha > \beta$ and a neighborhood W of x with $W \times \{\alpha\} \subset V$. Then $f \in (W \times (-\infty,\alpha))^- \subset V^-$, because epigraphs recede in the upward direction. □

LEMMA 2.2. *Let X be a normed linear space, and let $f \in \Gamma(X) \cap (K^C)^+$, where K is a weakly compact subset of $X \times R$. Then there is a finite subset $\{a_1, a_2, \ldots, a_n\}$ of $\operatorname{Aff}(X)$ and $\varepsilon > 0$ such that for each $i \leq n$,*

$$\inf_{x\in X} f(x) - a_i(x) > \varepsilon \quad \textit{and} \quad \sup a_i \in (K^C)^+.$$

PROOF. Let $k = (x_0,\alpha_0)$ be an arbitrary element of K. Since f is the supremum of the continuous affine functionals that it majorizes [**5**, p. 114], there exists $a_k \in \operatorname{Aff}(X)$ with

$$\inf_{x\in X} f(x) - a_k(x) > 0 \quad \text{and} \quad a_k(x_0) > \alpha_0.$$

The second condition means that k lies in the interior of the hypograph of a_k, a weakly open subset of $X \times R$ because a_k is continuous and affine. By the weak compactness of K, there exists $k(1), k(2), \ldots, k(n)$ in K with

$$K \subset \bigcup_{i=1}^{n} \operatorname{int}(\operatorname{hypo} a_{k(i)}).$$

Since for each $i \in \{1, 2, \ldots, n\}$ we have $\inf_{x \in X} f(x) - a_{k(i)}(x) > 0$, there exists $\varepsilon > 0$ such that for each $i \in \{1, 2, \ldots, n\}$ and each $x \in X$, $f(x) - a_{k(i)}(x) > \varepsilon$. The desired family of affine functionals is thus $\{a_{k(i)} : 1 \leq i \leq n\}$. □

Lemma 2.2 yields a topological version of the fact that a lower semicontinuous proper convex function is the supremum of the continuous affine functionals that it majorizes (see also §3.5.2 of [**1**]).

THEOREM 2.3. *Let X be a normed linear space, and let $f \in \Gamma(X)$. Let Ω be the finite subsets of* hypo f^*, *ordered by inclusion. For each $F = \{(y_1, \alpha_1), (y_2, \alpha_2), \ldots, (y_n, \alpha_n)\}$ in Ω, define $h_F \in \Gamma(X)$ by $h_F(x) = \sup_{1 \leq i \leq n} \langle y_i, x\rangle - \alpha_i$. Then $F \to h_F$ is an increasing net in $\Gamma(X)$ that is τ_M-convergent to f.*

We also require a general continuity result, which is surely known in some form. The proof is left to the reader.

LEMMA 2.4. *Let X be a normed linear space, and let $C_B(X, R)$ be the norm continuous real valued functions on X that are bounded on bounded subsets of X, equipped with the (locally convex metrizable) topology of uniform convergence on bounded subsets of X. Then if X^* is equipped with the norm topology, φ: $X^* \times R \to C_B(X, R)$ defined by $\varphi(y, \alpha)(x) = \langle y, x\rangle - \alpha$ is continuous.*

3. Results.

THEOREM 3.1. *Let X be a reflexive Banach space. Then the Young-Fenchel transform $f \to f^*$ is a homeomorphism of $\langle \Gamma(X), \tau_M\rangle$ onto $\langle \Gamma(X^*), \tau_M\rangle$.*

PROOF. Since $f \to f^*$ is an involution, it suffices to show that the transform $f^* \to f$ is continuous. To this end, we show that the inverse image of each subbasic open set in $\langle \Gamma(X), \tau_M\rangle$ is open in $\langle \Gamma(X^*), \tau_M\rangle$.

Suppose K is a weakly compact subset of $X \times R$, and $f \in (K^C)^+$, i.e., epi $f \cap K = \varnothing$. By Lemma 2.2, there exist continuous affine functionals $\{a_1, a_2, \ldots, a_n\}$ on X and $\varepsilon > 0$ such that for each $i \leq n$,

$$\inf_{x \in X} f(x) - a_i(x) > \varepsilon \quad \text{and} \quad \sup a_i \in (K^C)^+. \tag{1}$$

For each index i, let $y_i \in X^*$ and $\alpha_i \in R$ represent the affine function $a_i + \varepsilon$, in that for all $x \in X$,

$$a_i(x) + \varepsilon = \langle y_i, x\rangle - \alpha_i.$$

Since $\pi(K)$ is weakly compact and therefore norm bounded, by Lemma 2.4, for each index i there exists a norm neighborhood U_i of (y_i, α_i) such that for all $(y, \delta) \in U_i$ for all $x \in \pi(K)$, we have

$$|\langle y, x\rangle - \delta - (\langle y_i, x\rangle - \alpha_i)| < \varepsilon.$$

By equation (1), $(y_i, \alpha_i) \in \operatorname{epi} f^*$ for each $i \in \{1, 2, \ldots, n\}$, and it follows that $\bigcap_{i=1}^{n} U_i^-$ is a τ_M-neighborhood of f^* in $\Gamma(X^*)$. We claim that if $h^* \in \bigcap_{i=1}^{n} U_i^-$, then $h \in (K^C)^+$. To see that $\operatorname{epi} h$ does not meet K, we show that each point (x_0, α_0) of K is not in the epigraph of h. By the choice of the affine functionals $\{a_i : 1 \le i \le n\}$, there exists an index i such that $\alpha_0 < a_i(x_0)$. Choose $(y, \delta) \in \operatorname{epi} h^* \cap U_i$. This means that $\sup_{x \in X} h(x) - (\langle y, x \rangle - \delta) \ge 0$, and, in particular, $h(x_0) \ge \langle y, x_0 \rangle - \delta$. But by the choice of U_i, we have

$$\langle y, x_0 \rangle - \delta > \langle y_i, x_0 \rangle - \alpha_i - \varepsilon = a_i(x_0) > \alpha_0.$$

As a result, $h(x_0) > \alpha_0$, so that (x_0, α_0) fails to lie in $\operatorname{epi} h$. This proves that $h \in (K^C)^+$, provided $h^* \in \bigcup_{i=1}^n U_i^-$.

To complete the proof, by Lemma 2.1, it suffices to show that whenever W is a norm open subset of X, then the inverse image of $(W \times (-\infty, \beta))^-$ under $f^* \to f$ is τ_M-open in $\Gamma(X^*)$. Suppose $f \in (W \times (-\infty, \beta))^-$; this means that for some $x_0 \in W$, we have $f(x_0) < \beta$. Choose $\varepsilon \in (0, 1)$ such that $x_0 + \varepsilon B \subset W$. Also, pick a scalar μ satisfying

$$f(x_0) < \mu < \min\{\beta, f(x_0) + 1\}.$$

Since $\operatorname{epi} f = \operatorname{epi} f^{**}$, the choice of μ ensures

(i) $f^*(y) > \langle y, x_0 \rangle - \mu$ for each $y \in X^*$, and

(ii) there exists y_0 in X^* with $f^*(y_0) < \langle y_0, x_0 \rangle - \mu + 1$.

Denote the affine functional $y \to \langle y, x_0 \rangle - \mu$ on X^* by a_0. Also, let $\lambda = \max\{\|y_0\|, 4/\varepsilon\}$, and let K be that part of the graph of a_0 within the vertical cylinder $\{(y, \alpha) : y \in y_0 + \lambda B^*\}$. As K is the intersection of two closed convex sets (the graph and the cylinder), K is a closed convex set, and since a_0 is Lipschitz, its graph restricted to $y_0 + \lambda B^*$ is a bounded subset of $X^* \times R$. Thus, K is a weakly compact subset of the reflexive space $X^* \times R$ disjoint from $\operatorname{epi} f^*$. By (ii), there is an open neighborhood U of y_0 contained in $y_0 + B^*$ such that for each $y \in U$, we have

$$f^*(y_0) - 1 < \langle y, x_0 \rangle - \mu. \tag{2}$$

Clearly,

$$f^* \in (K^C)^+ \cap (U \times (-\infty, f^*(y_0) + 1))^-.$$

We show that if $h^* \in (K^C)^+ \cap (U \times (-\infty, f^*(y_0) + 1))^-$, then $h \in (W \times (-\infty, \beta))^-$. Choose $y_1 \in U$ with $h^*(y_1) < f^*(y_0) + 1$. Since K is a weakly compact convex set disjoint from the closed convex set $\operatorname{epi} h^*$, the sets K and $\operatorname{epi} h^*$ are a positive distance apart, whence K and $\operatorname{epi} h^*$ can be strongly separated by a closed hyperplane in $X^* \times R$. This hyperplane is not vertical, for since $\lambda > 4$ and $\|y_0 - y_1\| \le 1$, we have $y_1 \in \pi(K) \cap \pi(\operatorname{epi} h)$. Thus, the hyperplane is the graph of a continuous affine functional on X^*, say, $a(y) = \langle y, x \rangle - \alpha$. Since h^* majorizes a, $(x, \alpha) \in \operatorname{epi} h^{**} = \operatorname{epi} h$.

We claim that $(x, \alpha) \in W \times (-\infty, \beta)$. The idea of the proof is twofold: First, since $\pi(K)$ contains the origin of X^*, we have $(\theta^*, -\mu) \in K$. The point $(\theta^*, -\mu)$ must thus lie below $(\theta^*, a(\theta^*))$, so that $\alpha < \mu$. Second, the gap between $\operatorname{epi} h^*$ and K at a point near the center of the disc K is narrow relative to the width of K, so that the graph of the affine function a must be nearly parallel to the graph of a_0. Analytically, this means that $\|x - x_0\|$ is small. The details now follow.

Since $\lambda \ge \|y_0\|$, we see that $\theta^* \in y_0 + \lambda B^*$, whence $(\theta^*, a_0(\theta^*)) = (\theta^*, -\mu) \in K$. As a result, $a(\theta^*) = -\alpha$ must exceed $-\mu$, so that $\alpha < \mu < \beta$. We intend to show that $\|x - x_0\| \le \varepsilon$. Suppose, to the contrary that $\|x - x_0\| > \varepsilon$. By reflexivity, there is direction of steepest descent for the functional $x - x_0$, i.e., a unit vector $w \in X^*$ with $\langle w, x - x_0 \rangle = -\|x - x_0\|$. Set $y_2 = (y_1 + (\lambda/2)w)$. Since $\lambda \geq 4/\varepsilon > 4$, we have

$$\|y_0 - y_2\| \leq \|y_0 - y_1\| + \lambda/2 < 1 + \lambda/2 < \lambda,$$

so that y_2 lies in $\pi(K)$. We show $(y_2, a(y_2))$ lies below K, contradicting the (strong) separation of K from epi h^* by the graph of a. We have

$$\begin{aligned} a(y_2) - a_0(y_2) &= \langle y_1 + (\lambda/2)w, x\rangle - \alpha - \langle y_1 + (\lambda/2)w, x_0\rangle + \mu \\ &= \langle (\lambda/2)w, x - x_0\rangle + \langle y_1, x - x_0\rangle - \alpha + \mu \\ &= -(\lambda/2)\|x - x_0\| + a(y_1) - \langle y_1, x_0\rangle + \mu. \end{aligned} \tag{3}$$

Since h^* majorizes a, and $y_1 \in U$ and $h^*(y_1) \in (-\infty, f^*(y_0) + 1)$, inequality (2) yields

$$a(y_1) \leq h^*(y_1) < f^*(y_0) + 1 < \langle y_1, x_0\rangle - \mu + 2. \tag{4}$$

By assumption, $\|x - x_0\| > \varepsilon$, and since $\lambda > 4/\varepsilon$, formulas (3) and (4) together yield

$$a(y_2) - a_0(y_2) < (-1/2)(4/\varepsilon)\varepsilon + 2 = 0.$$

Having obtained the desired contradiction, we conclude that $\|x - x_0\| \leq \varepsilon$, so that

$$(x, \alpha) \in \operatorname{epi} h \cap ((x_0 + \varepsilon B) \times (-\infty, \beta)) \subset \operatorname{epi} h \cap (W \times (-\infty, \beta)),$$

completing the proof of the continuity of the Young-Fenchel transform. □

Using his sequential continuity theorem, Mosco established the sequential continuity of the polar operation from $\mathscr{C}(X)$ to $\mathscr{C}(X^*)$ by showing that Mosco convergence of a sequence in $\Gamma(X)$ ensures Mosco convergence of sublevel sets above a minimal height. It seems worthwhile to extend his polar result to a legitimate continuity theorem. The geometrical simplicity of the proof is indeed startling.

LEMMA 3.2. *Let X be a reflexive Banach space, and suppose $\langle f_\lambda \rangle$ is a net in $\Gamma(X)$ τ_M-convergent to $f \in \Gamma(X)$. Then for each $\alpha > \inf f$, we have* $\operatorname{sub}(f;\alpha) = \tau_M\text{-}\lim \operatorname{sub}(f_\lambda;\alpha)$.

PROOF. Let $\alpha > \inf f$ be fixed. Suppose W is open in X and $\operatorname{sub}(f;\alpha) \in W^-$. Choose $x \in W$ with $f(x) \leq \alpha$. Since $\alpha > \inf f$, there exists $x_1 \in X$ with $f(x_1) < \alpha$. Since the line segment joining $(x, f(x))$ to $(x_1, f(x_1))$ lies in epi f, it must meet $W \times (-\infty, \alpha)$. As a result, $\langle \operatorname{epi} f_\lambda \rangle$ must meet $W \times (-\infty, \alpha)$ eventually, which ensures that $\langle \operatorname{sub}(f_\lambda;\alpha)\rangle$ meets W eventually. Suppose now that $\operatorname{sub}(f;\alpha) \cap K = \varnothing$, where K is a weakly compact subset of X. This is equivalent to saying that

$$\operatorname{epi} f \cap (K \times \{\alpha\}) = \varnothing.$$

But $K \times \{\alpha\}$ is a weakly compact subset of $X \times R$; so, by τ_M-convergence of $\langle f_\lambda \rangle$ to f, we must have

$$\operatorname{epi} f_\lambda \cap (K \times \{\alpha\}) = \varnothing$$

eventually. Thus, eventually, $\operatorname{sub}(f_\lambda;\alpha) \cap K = \varnothing$. □

THEOREM 3.3. *Let X be a reflexive Banach space. Then the polar map $C \to C^\circ$ is a continuous function from $\langle \mathscr{C}(X), \tau_M \rangle$ to $\langle \mathscr{C}(X^*), \tau_M \rangle$.*

PROOF. Evidently, $C \to I(\cdot, C)$ is an embedding of $\langle \mathscr{C}(X), \tau_M \rangle$ into $\langle \Gamma(X), \tau_M \rangle$, whence by Theorem 3.1, $C \to I^*(\cdot, C)$ is an embedding of $\langle \mathscr{C}(X), \tau_M \rangle$ into

$\langle \Gamma(X^*), \tau_M \rangle$. But for each $C \in \mathscr{C}(X)$, $I^*(\cdot, C)$ is the support functional $s(\cdot, C)$ for C [**6**, §14], and

$$\inf_{y \in X^*} s(y, C) \leq s(\theta^*, C) = 0 < 1.$$

Suppose now $C_1 \in \mathscr{C}(X)$ is fixed, and $\langle \mathscr{C}_\lambda \rangle$ is a net in $\mathscr{C}(X)$ τ_M-convergent to C_1. By Lemma 3.2 and the above remarks, we have

$$C_1^\circ = \mathrm{sub}(s(\cdot, C_1); 1) = \tau_M\text{-}\lim[\mathrm{sub}(s(\cdot, C_\lambda); 1)] = \tau_M\text{-}\lim C_\lambda^\circ.$$

This establishes τ_M-continuity of the polar operation. □

All results involving the Young-Fenchel transform untimately rest on the correspondence between the points of the epigraph of f^* for a proper lower semicontinuous convex function f and the continuous affine functionals majorized by f, given by $(y, \alpha) \to a(y, \alpha)$, where

$$a(y, \alpha)(x) = \langle y, x \rangle - \alpha \qquad (x \in X).$$

Without reflexivity, or even completeness, this is a *continuous* parametrization. With reflexivity, much more is true.

THEOREM 3.4. *Let X be a normed linear space, and let $\psi\colon X^* \times R \to \mathrm{Aff}(X)$ be defined by $\psi(y, \alpha) = a(y, \alpha)$. Then ψ is continuous, where X^* is equipped with the norm topology and* $\mathrm{Aff}(X)$ *is equipped with the Mosco topology. Moreover, if X is reflexive, then ψ is a homeomorphism.*

PROOF. Suppose $a(y_0, \alpha_0) \in (K^C)^+$ where K is a weakly compact subset of $X \times R$. Define $h\colon \pi(K) \to R$ by $h(x) = \max\{\beta\colon (x, \beta) \in K\}$. It is easy to check that h is weakly upper semicontinuous on K. Since $a(y_0, \alpha_0)(x) > h(x)$ for each x in K, and $a(y_0, \alpha_0) - h$ is weakly lower semicontinuous, by the weak compactness of $\pi(K)$, $a(y_0, \alpha_0) - h$ achieves a minimum positive value on $\pi(K)$. Thus, there exists $\varepsilon > 0$ such that for each $x \in \pi(K)$,

$$a(y_0, \alpha_0)(x) > h(x) + \varepsilon.$$

Since $\pi(K)$ is weakly compact, it is weakly bounded and is thus norm bounded [**5**, p. 74]. Applying Lemma 2.4, we see that there exists a norm neighborhood U of (y_0, α_0) such that for each $(y, \alpha) \in U$ and each $x \in \pi(K)$, we have $a(y, \alpha)(x) > h(x)$. This means that $a(y, \alpha) \in (K^C)^+$.

Next suppose $a(y_0, \alpha_0) \in (W \times (-\infty, \beta))^-$ for some open subset W of X. There exists $x_0 \in W$ with $a(y_0, \alpha_0)(x_0) < \beta$. Pick $\varepsilon > 0$ such that

$$a(y_0, \alpha_0)(x_0) + \varepsilon < \beta. \tag{5}$$

If $x_0 = \theta$, then $a(y_0, \alpha_0)(x_0) = -\alpha_0$. As a result, if $|\alpha - \alpha_0| < \varepsilon$ and y is arbitrary, then by (5),

$$a(y, \alpha)(x_0) = a(y, \alpha)(\theta) = -\alpha < \beta$$

and we have $a(y, \alpha) \in (W \times (-\infty, \beta))^-$. If $x_0 \neq \theta$, we claim that

$$V \equiv (y_0 + (\varepsilon/2\|x_0\|)B^*) \times (\alpha_0 - \varepsilon/2, \alpha_0 + \varepsilon/2)$$

is mapped by ψ into $(W \times (-\infty, \beta))^-$. Suppose $(y, \alpha) \in V$. We have

$$|a(y, \alpha)(x_0) - a(y_0, \alpha_0)(x_0)| \leq \|y - y_0\| \cdot \|x_0\| + |\alpha - \alpha_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

As a consequence of (5), $a(y,\alpha)(x_0) < \beta$, and $a(y,\alpha) \in (W \times (-\infty,\beta))^-$, because $(x_0, a(y,\alpha)(x_0)) \in W \times (-\infty,\beta)$. Continuity of ψ is now established, without reflexivity.

We now show that $\psi^{-1}\colon \mathrm{Aff}(X) \to X^* \times R$ is continuous, provided X is reflexive. This is almost immediate from Theorem 3.1. Fix (y_0, α_0) in $X^* \times R$, and suppose $\langle (y_\lambda, \alpha_\lambda) \rangle$ is a net in $X^* \times R$ such that

$$a(y_0, \alpha_0) = \tau_M\text{-}\lim a(y_\lambda, \alpha_\lambda).$$

By Theorem 3.1, $a^*(y_0, \alpha_0) = \tau_M\text{-}\lim a^*(y_\lambda, \alpha_\lambda)$. In terms of epigraphs, this says that $\{(y_0, \beta)\colon \beta \geq \alpha_0\} = \tau_M\text{-}\lim\{(y_\lambda, \beta)\colon \beta \geq \alpha_\lambda\}$. It is now obvious that $\lim\|y_\lambda - y_0\| = 0$ and $\lim \alpha_\lambda = \alpha_0$. □

COROLLARY 3.5. *Let X be a reflexive Banach space, and let f be a proper lower semicontinuous convex function on X. Then* epi f^* *as a subspace of $X^* \times R$ with the norm topology, is homeomorphic to the continuous affine functions majorized by f, equipped with the Mosco topology.*

COROLLARY 3.6. *Let X be a reflexive Banach space. Then $\langle \mathrm{Aff}(X), \tau_M \rangle$ is completely metrizable.*

To conclude, we show that when X is separable and reflexive, we can select for each $f \in \Gamma(X)$ a continuous affine function a_f majorized by f in such a manner that $f \to a_f$ is a continuous function on $\langle \Gamma(X), \tau_M \rangle$.

THEOREM 3.7. *Let X be a separable reflexive Banach space. Let $f_1 \in \Gamma(X)$ and let a_1 be a fixed continuous affine function on X majorized by f_1. Then there exists $\sigma\colon \langle \Gamma(X), \tau_M \rangle \to \langle \mathrm{Aff}(X), \tau_M \rangle$ such that σ is continuous, $\sigma(f_1) = a_1$, and for each $f \in \Gamma(X)$ and $x \in X$, we have $\sigma(f)(x) \leq f(x)$.*

PROOF. Suppose $f \in \Gamma(X)$ is arbitrary. By Theorem 3.1, whenever V is an open subset of $X^* \times R$ that meets the epigraph of f^*, there is a τ_M-neighborhood of f such that $\mathrm{epi}\, h \cap V \neq \varnothing$ for each h in the neighborhood. Thus, if we view $f \to \mathrm{epi}\, f^*$ as a set valued function from $\Gamma(X)$ to $\mathscr{C}(X^* \times R)$, then this correspondence is lower semicontinuous in the sense of Kuratowski ([**8**, p. 73; **9**, p. 173 and **11**]). Clearly, the correspondence has closed convex values. If $a_1(x) = \langle y_1, x \rangle - \alpha_1$, and we assign $\{(y_1, \alpha_1)\}$ (rather than $\mathrm{epi}\, f_1^*$) to f_1, the correspondence remains lower semicontinuous and convex valued. By Theorem 4.3 of [**2**], $\langle \Gamma(X), \tau_M \rangle$ is metrizable and is thus paracompact. Applying Michael's selection theorem [**11**], there exists a continuous function $\rho\colon \langle \Gamma(X), \tau_M \rangle \to X^* \times R$ such that for each $f \in \Gamma(X)$, we have both $\rho(f) \in \mathrm{epi}\, f^*$ and $\rho(f_1) = (y_1, \alpha_1)$. With $\psi\colon X^* \times R \to \mathrm{Aff}(X)$ as in the statement of Theorem 3.4, $\sigma = \psi \circ \rho$ is the desired function. □

Separability of X is only used to guarantee paracompactness of the function space $\langle \Gamma(X), \tau_M \rangle$. We have no idea whether or not $\langle \Gamma(X), \tau_M \rangle$ is paracompact for a nonseparable reflexive space X.

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