FIXED POINTS FOR DISCONTINUOUS QUASI-MONOTONE MAPS IN $\mathbb{R}^n$

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Abstract. Let $K^n$ be the unit cube in $\mathbb{R}^n$ and $f = (f_1, f_2, \ldots, f_n): K^n \to \mathbb{R}^n$. It is known that $f$ has maximal and minimal fixed points in $K^n$ if $f: K^n \to K^n$ and $f$ is monotone increasing. In this paper, a weaker condition, namely quasi-monotonicity, is considered and it is proved that the above mentioned conclusion is still true if $f$ is either quasi-monotone and

$$\liminf_{t \to 0} \frac{[f_i(x + te_i) - f_i(x)]}{t} \neq -\infty,$$

or $-f$ is quasi-monotone and

$$\limsup_{t \to 0} \frac{[f_i(x + te_i) - f_i(x)]}{t} \neq +\infty.$$

1. Introduction. Let $(X, P)$ be an ordered Banach space with a positive cone $P \subset X$ (see [1] and [4]), $u_0, v_0 \in X$ with $u_0 \leq v_0$ and $F: [u_0, v_0] \to X$ be increasing (i.e., $F_x \leq F_y$ for $x \leq y$). Then it is well known that $F$ has both maximal and minimal fixed points in $[u_0, v_0]$ if $u_0 \leq F u_0, F v_0 \leq v_0$ and every chain has a least upper bound in $X$. The same result is evidently true if we replace monotonicity by

$$(1.1) \quad F y - F x \geq -\alpha(y - x) \quad \text{for all } y \geq x \text{ and some } \alpha \geq 0$$

(see [5]). In case $X = \mathbb{R}$ and $P = \mathbb{R}^+$ (1.1) implies that

$$\liminf_{y \to x} \frac{F(y) - F(x)}{y - x} \neq -\infty.$$

It is this simple observation which leads to the present paper. In fact, we can improve condition (1.1) considerably in this direction in case $X = \mathbb{R}^n$.

We recall that $F: X \to X$ is said to be quasi-monotone (increasing) if $y \geq x$, $x^* \in P^*$ and $x^*(y-x) = 0$ imply $x^*(F y - F x) \geq 0$; where $P^* = \{x^* \in X^*: x^*(x) > 0$ for all $x \in P\}$. In case $X = \mathbb{R}^n$ and $P = \mathbb{R}^n_+$, $f = (f_1, \ldots, f_n): \mathbb{R}^n \to \mathbb{R}^n$ is quasi-monotone iff $f_i(x_1, \ldots, x_n)$ is nondecreasing in $x_j, j \neq i$, for every $i$. For an $f: \mathbb{R} \to \mathbb{R}$, we define $\overline{f}(x + 0)$, the upper-right limit, to be $\limsup_{y \to x + 0} f(y)$. Similarly, we define $\underline{f}(x + 0)$, $\overline{f}(x - 0)$ and $\underline{f}(x - 0)$, the lower-right, upper-left and lower-left limits, respectively. For $f: \mathbb{R}^n \to \mathbb{R}^n$ and $\{e_1, e_2, \ldots, e_n\}$, the standard base for $\mathbb{R}^n$, we let $D^+ f_i(x)$, the upper-right Dini derivative about $i$ to be

$$D^+ f_i(x) = \limsup_{t \to 0^+} \frac{[f_i(x + te_i) - f_i(x)]}{t}.$$
Analogously, we define $D^+_i(x)$, $D^-_i(x)$, and $D^+_i(x)$ the lower-right, upper-left and lower-left Dini derivatives about $i$. We also let

$$K_n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ for every } i\}.$$ 

As a preparation, we first consider the case $X = \mathbb{R}$ in the following section.

2. The case $X = \mathbb{R}$. Let $J = [0,1]$ and consider the following conditions for $f: J \to \mathbb{R}$

(2.1) \[ \min\{D^+_f(x), D^-_f(x)\} > -\infty \quad \text{on } J, \]

(2.2) \[ \max\{D^+_f(x), D^-_f(x)\} < +\infty \quad \text{on } J, \]

where only right Dini derivatives are considered at $x = 0$, only left ones at $x = 1$. Then we have

**Theorem 1.** Let $f: J \to \mathbb{R}$ satisfy (2.1) and $0 < f(0), f(1) < 1$. Then $f$ has a maximal and a minimal fixed point in $J$.

**Proof.** We have $0 \in A = \{x \in J : x \leq f(x)\}$. Let $x^* = \sup A$. Then $D^-_f(x^*) > -\infty$ implies $x^* \in A$. We also have $1 \in B = \{x \in [x^*,1] : f(x) \leq x\}$. Let $y_* = \inf B$. Then $D^+_f(y_*) > -\infty$ implies $y_* \in B$. We have $x^* = y_*$ since otherwise $x^* < z = (y_* + x^*)/2 < y_*$ and neither $z \leq f(z)$ nor $f(z) \leq z$ is possible. Therefore $x^* \in C = \{x \in J : f(x) = x\}$. Let $u = \inf C$ and $v = \sup C$. Then a repetition of the above arguments shows that $u \in C$ and $v \in C$, hence $u, v$ are the minimal and maximal fixed points of $f$. Q.E.D.

**Corollary 1.** Let $f: J \to \mathbb{R}$ satisfy (2.2) and $f(0) \leq 0$, $1 \leq f(1)$. Then $f$ has a maximal and a minimal fixed point in $J$.

This follows by application of Theorem 1 to $g(x) = (f(x) - \alpha x)/(1 - \alpha)$ with $\alpha > 1$.

**Example.**

$$f_1(x) = \begin{cases} 1 - \sqrt{x}, & x \in (0,1] \\ \frac{1}{2}, & x = 0 \end{cases}$$

satisfies (2.1) but not (1.1). This indicates that (2.1) is weaker than (1.1).

**Remark.** Condition (2.1) or (2.2) implies that $f(x)$ is continuous in $J - N$ with $N$ at most countable. This is because, for example, (2.1) implies

$$\overline{f}(x-0) \leq f(x) \leq \overline{f}(x+0)$$

at every $x \in [0,1]$, while W. H. Young’s Theorem [3, p. 304] states that, for any $f: \mathbb{R} \to \mathbb{R}$,

$$\overline{f}(x+0) = f(x-0) \leq f(x) \leq \overline{f}(x+0) = \overline{f}(x-0)$$

at every $x$ except at points of an enumerable set.

3. The general case $X = \mathbb{R}^n$. For $f: K_n \to \mathbb{R}^n$ we consider the following conditions

(3.1) \[ \min\{D^+_i(x), D^-_i(x)\} > -\infty \quad \text{for all } x \in K_n \text{ and } 1 \leq i \leq n, \]

(3.2) \[ \max\{D^+_i(x), D^-_i(x)\} < +\infty \quad \text{for all } x \in K_n \text{ and } 1 \leq i \leq n, \]
An immediate attempt to generalize Theorem 1 is to replace (2.1) by (3.1). However, the following counter-example shows that even condition (3.3) (stronger than (3.1)) is not enough, where

\[
(3.3) \quad \liminf_{t \to 0} \frac{f_i(x + t(e_j) - f_i(x)}{t} > -\infty \quad \text{for all } x \in K_n \text{ and } 1 \leq i, j \leq n.
\]

**Counterexample.** Let \(X = \mathbb{R}^2\) and define \(f(x, y) = (f_1(y), f_2(x))\) with \(f_1 = x[1/2, 1]\) and \(f_2(x) = 1 - x\) on \(J\). Then \(f = (f_1, f_2): K_2 \to \mathbb{R}^2\) certainly satisfies the requirements proposed above and furthermore, \(f_1(x, y)\) is increasing in \(y\) and \(f_2(x, y)\) is linear. But \(f\) does not have a fixed point since \(f_1(f_2(x)) = x\) has no solution in \(J\).

It is also clear that a quasi-monotone map \(f: K_n \to \mathbb{R}^n\) (even \(f: K_n \to K_n\)) may not have a fixed point. The following theorem shows that a combination of quasi-monotonicity with one of the conditions (3.1) and (3.2) is good enough to guarantee the existence of fixed points.

**Theorem 2.** Let \(f: K_n \to \mathbb{R}^n\) be quasi-monotone, satisfy (3.1). Furthermore, suppose that

\[
0 < f_i(x - x_i e_i) \quad \text{and} \quad f_i(x + (1 - x_i) e_i) < 1 \quad \text{on } K_n
\]

for all \(1 \leq i \leq n\). Then \(f\) has a maximal and a minimal fixed point in \(K_n\).

**Proof.** For \(x \in \mathbb{R}^n\) and \(1 \leq k \leq n - 1\) let \(x^k = (x_{k+1}, \ldots, x_n)\). Consider

\[
A_1(x^1) = \{t \in J : t < f_1(t, x^1)\} \quad \text{and} \quad T_1(x^1) = \sup A_1(x^1).
\]

By the proof of Theorem 1 we have

\[
T_1(x^1) \in A_1(x^1) \quad \text{and even} \quad T_1(x^1) = f_1(T_1(x^1), x^1).
\]

Since \(f\) is quasi-monotone, \(x^1 \leq y^1\) implies \(T_1(x^1) \in A_1(y^1)\) and therefore \(T_1(x^1) \leq T_1(y^1)\). Now let

\[
A_2(x^2) = \{t \in J : t < f_2(T_1(t, x^2), t, x^2)\} \quad \text{and} \quad T_2(x^2) = \sup A_2(x^2).
\]

Since \(T_1\) is increasing, (3.1) yields (2.1) for \(f_2(T_1(\cdot, x^2), \cdot, x^2)\), hence

\[
T_2(x^2) \in A_2(x^2) \quad \text{and} \quad T_2(x^2) = f_2(T_1(T_2(x^2), x^2), T_2(x^2), x^2).
\]

By repeating this argument we finally get a fixed point \(x^*\) of \(f\) with

\[
x_{n-1}^* = T_{n-1}(x_{n-1}^*), \quad x_{n-2}^* = T_{n-2}(x_{n-1}^*, x_{n-2}^*), \quad \text{and so on.}
\]

Now suppose that \(y\) is another fixed point of \(f\) in \(K_n\). Then

\[
y_1 \leq T_1(y^1), \quad \text{hence} \quad y_2 \leq f_2(T_1(y_2, y_2^2), y_2, y_2^2)
\]

and therefore \(y_2 \leq T_2(y^2)\). Proceeding this way we finally get \(y_n \leq x_n^*\), hence \(y \leq x^*\), and therefore \(x^*\) is the maximal fixed point of \(f\) in \(K_n\).

If we let \(B_1(x^1) = \{t \in J : f_1(t, x^1) \leq t\}\) and \(S_1(x^1) = \inf B_1(x^1)\), then it is clear that the same procedure yields the minimal fixed point of \(f\). Q.E.D.

Since Corollary 1 can also be proved by defining first \(A = \{x \in J : x \geq f(x)\}\), and then proving \(x^* = \sup A \in A\), analogous to the proof of Theorem 2 we now have
COROLLARY 2. Let $f: K_n \to \mathbb{R}^n$ be such that (3.2) is satisfied, $-f$ is quasi-monotone and
\[ f_i(x - x_i e_i) \leq 0 \quad \text{and} \quad 1 \leq f_i(x + (1 - x_i)e_i) \quad \text{on } K_n \]
for all $1 \leq i \leq n$. Then $f$ has maximal and minimal fixed points in $K_n$.

REMARK. It remains open whether the results in this paper can be extended to the case $X = l^\infty$ or $l^p$ for some $p \geq 1$.

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REFERENCES


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