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FIXED POINTS FOR DISCONTINUOUS QUASI-MONOTONE MAPS IN $\mathbf{R}^n$

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ABSTRACT. Let K_n be the unit cube in R^n and $f = (f_1, f_2, \dots, f_n)\colon K_n \to R^n$. It is known that f has maximal and minimal fixed points in K^n if $f\colon K_n \to K_n$ and f is monotone increasing. In this paper, a weaker condition, namely quasi-monotonicity, is considered and it is proved that the above mentioned conclusion is still true if f is either quasi-monotone and

$$\liminf_{t\to 0} \frac{[f_i(x+te_i) - f_i(x)]}{t} \neq -\infty,$$

or $-f$ is quasi-monotone and

$$\limsup_{t\to 0} \frac{[f_i(x+te_i) - f_i(x)]}{t} \neq +\infty.$$

1. Introduction. Let (X, P) be an ordered Banach space with a positive cone $P \subset X$ (see [**1**] and [**4**]), u_0, $v_0 \in X$ with $u_0 \leq v_0$ and $F\colon [u_0, v_0] \to X$ be increasing (i.e., $Fx \leq Fy$ for $x \leq y$). Then it is well known that F has both maximal and minimal fixed points in $[u_0, v_0]$ if $u_0 \leq Fu_0$, $Fv_0 \leq v_0$ and every chain has a least upper bound in X. The same result is evidently true if we replace monotonicity by

$$(1.1) \qquad Fy - Fx \geq -\alpha(y-x) \quad \text{for all } y \geq x \text{ and some } \alpha \geq 0$$

(see [**5**]). In case $X = \mathbf{R}$ and $P = \mathbf{R}^+$ (1.1) implies that

$$\liminf_{y\to x} \frac{F(y) - F(x)}{y-x} \neq -\infty.$$

It is this simple observation which leads to the present paper. In fact, we can improve condition (1.1) considerably in this direction in case $X = R^n$.

We recall that $F\colon X \to X$ is said to be quasi-monotone (increasing) if $y \geq x$, $x^* \in P^*$ and $x^*(y-x) = 0$ imply $x^*(Fy - Fx) \geq 0$; where $P^* = \{x^* \in X^*\colon x^*(x) \geq 0$ for all $x \in P\}$. In case $X = \mathbf{R}^n$ and $P = \mathbf{R}^n_+$, $f = (f_1, \dots, f_n)\colon \mathbf{R}^n \to \mathbf{R}^n$ is quasi-monotone iff $f_i(x_1, \dots, x_n)$ is nondecreasing in $x_j, j \neq i$, for every i. For an $f\colon \mathbf{R} \to \mathbf{R}$, we define $\overline{f}(x+0)$, the upper-right limit, to be $\limsup_{y\to x+0} f(y)$. Similarly, we define $\underline{f}(x+0)$, $\overline{f}(x-0)$ and $\underline{f}(x-0)$, the lower-right, upper-left and lower-left limits, respectively. For $f\colon \mathbf{R}^n \to \mathbf{R}^n$ and $\{e_1, e_2, \dots, e_n\}$, the standard base for $\mathbf{R}^n$, we let $D^+f_i(x)$, the upper-right Dini derivative about i to be

$$D^+f_i(x) = \limsup_{t\to 0+} \frac{[f_i(x+te_i) - f_i(x)]}{t}.$$

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Analogously, we define $D_+ f_i(x)$, $D^- f_i(x)$, and $D_- f_i(x)$ the lower-right, upper-left and lower-left Dini derivatives about i. We also let

$$K_n = \{x \in \mathbf{R}^n : 0 \le x_i \le 1 \text{ for every } i\}.$$

As a preparation, we first consider the case $X = \mathbf{R}$ in the following section.

2. The case $X = \mathbf{R}$. Let $J = [0,1]$ and consider the following conditions for $f \colon J \to \mathbf{R}$

$$\min\{D_+ f(x), D_- f(x)\} > -\infty \quad \text{on } J, \tag{2.1}$$

$$\max\{D^+ f(x), D^- f(x)\} < +\infty \quad \text{on } J, \tag{2.2}$$

where only right Dini derivatives are considered at $x = 0$, only left ones at $x = 1$. Then we have

THEOREM 1. *Let $f \colon J \to \mathbf{R}$ satisfy* (2.1) *and* $0 \le f(0), f(1) \le 1$. *Then f has a maximal and a minimal fixed point in J.*

PROOF. We have $0 \in A = \{x \in J : x \le f(x)\}$. Let $x^* = \sup A$. Then $D_- f(x^*) > -\infty$ implies $x^* \in A$. We also have $1 \in B = \{x \in [x^*, 1] : f(x) \le x\}$. Let $y_* = \inf B$. Then $D_+ f(y_*) > -\infty$ implies $y_* \in B$. We have $x^* = y_*$ since otherwise $x^* < z = (y_* + x^*)/2 < y_*$ and neither $z \le f(z)$ nor $f(z) \le z$ is possible. Therefore $x^* \in C = \{x \in J : f(x) = x\}$. Let $u = \inf C$ and $v = \sup C$. Then a repetition of the above arguments shows that $u \in C$ and $v \in C$, hence u, v are the minimal and maximal fixed points of f. Q.E.D.

COROLLARY 1. *Let $f \colon J \to \mathbf{R}$ satisfy* (2.2) *and* $f(0) \le 0$, $1 \le f(1)$. *Then f has a maximal and a minimal fixed point in J.*

This follows by application of Theorem 1 to $g(x) = (f(x) - \alpha x)/(1 - \alpha)$ with $\alpha > 1$.

EXAMPLE.

$$f_1(x) = \begin{cases} 1 - \sqrt{x}, & x \in (0,1] \\ \frac{1}{2}, & x = 0 \end{cases}$$

satisfies (2.1) but not (1.1). This indicates that (2.1) is weaker than (1.1).

REMARK. Condition (2.1) or (2.2) implies that $f(x)$ is continuous in $J - N$ with N at most countable. This is because, for example, (2.1) implies

$$\overline{f}(x - 0) \le f(x) \le \underline{f}(x + 0)$$

at every $x \in [0,1]$, while W. H. Young's Theorem [**3**, p. 304] states that, for any $f \colon \mathbf{R} \to \mathbf{R}$,

$$\underline{f}(x + 0) = \underline{f}(x - 0) \le f(x) \le \overline{f}(x + 0) = \overline{f}(x - 0)$$

at every x except at points of an enumerable set.

3. The general case $X = R^n$. For $f \colon K_n \to \mathbf{R}^n$ we consider the following conditions

$$\min\{D_+ f_i(x), D_- f_i(x)\} > -\infty \quad \text{for all } x \in K_n \text{ and } 1 \le i \le n, \tag{3.1}$$

$$\max\{D^+ f_i(x), D^- f_i(x)\} < +\infty \quad \text{for all } x \in K_n \text{ and } 1 \le i \le n, \tag{3.2}$$

An immediate attempt to generalize Theorem 1 is to replace (2.1) by (3.1). However, the following counter-example shows that even condition (3.3) (stronger than (3.1)) is not enough, where

$$\liminf_{t\to 0} \frac{[f_i(x+t(e_j)-f_i(x)]}{t} > -\infty \quad \text{for all } x \in K_n \text{ and } 1 \le i,\ j \le n. \tag{3.3}$$

COUNTEREXAMPLE. Let $X = \mathbf{R}^2$ and define $f(x,y) = (f_1(y),\ f_2(x))$ with $f_1 = \chi_{[1/2,1]}$ and $f_2(x) = 1 - x$ on J. Then $f = (f_1, f_2)\colon K_2 \to \mathbf{R}^2$ certainly satisfies the requirements proposed above and furthermore, $f_1(x,y)$ is increasing in y and $f_2(x,y)$ is linear. But f does not have a fixed point since $f_1(f_2(x)) = x$ has no solution in J.

It is also clear that a quasi-monotone map $f\colon K_n \to R^n$ (even $f\colon K_n \to K_n$) may not have a fixed point. The following theorem shows that a combination of quasi-monotonicity with one of the conditions (3.1) and (3.2) is good enough to guarantee the existence of fixed points.

THEOREM 2. *Let $f\colon K_n \to \mathbf{R}^n$ be quasi-monotone, satisfy* (3.1). *Furthermore, suppose that*

$$0 \le f_i(x - x_i e_i) \quad \text{and} \quad f_i(x + (1-x_i)e_i) \le 1 \quad \text{on } K_n$$

for all $1 \le i \le n$. Then f has a maximal and a minimal fixed point in K_n.

PROOF. For $x \in \mathbf{R}^n$ and $1 \le k \le n-1$ let $x^k = (x_{k+1}, \dots, x_n)$. Consider

$$A_1(x^1) = \{t \in J : t \le f_1(t, x^1)\} \quad \text{and} \quad T_1(x^1) = \sup A_1(x^1).$$

By the proof of Theorem 1 we have

$$T_1(x^1) \in A_1(x^1) \quad \text{and even} \quad T_1(x^1) = f_1(T_1(x^1), x^1).$$

Since f is quasi-monotone, $x^1 \le y^1$ implies $T_1(x^1) \in A_1(y^1)$ and therefore $T_1(x^1) \le T_1(y^1)$. Now let

$$A_2(x^2) = \{t \in J : t \le f_2(T_1(t, x^2), t, x^2) \quad \text{and} \quad T_2(x^2) = \sup A_2(x^2).$$

Since T_1 is increasing, (3.1) yields (2.1) for $f_2(T_1(\cdot, x^2), \cdot, x^2)$, hence

$$T_2(x^2) \in A_2(x^2) \quad \text{and} \quad T_2(x^2) = f_2(T_1(T_2(x^2), x^2), T_2(x^2), x^2).$$

By repeating this argument we finally get a fixed point x^* of f with

$$x^*_{n-1} = T_{n-1}(x^*_n), \qquad x^*_{n-2} = T_{n-2}(x^*_{n-1}, x^*_n), \text{ and so on.}$$

Now suppose that y is another fixed point of f in K_n. Then

$$y_1 \le T_1(y^1), \quad \text{hence } y_2 \le f_2(T_1(y_2, y^2), y_2, y^2)$$

and therefore $y_2 \le T_2(y^2)$. Proceeding this way we finally get $y_n \le x^*_n$, hence $y \le x^*$, and therefore x^* is the maximal fixed point of f in K_n.

If we let $B_1(x^1) = \{t \in J\colon f_1(t, x^1) \le t\}$ and $S_1(x^1) = \inf B_1(x^1)$, then it is clear that the same procedure yields the minimal fixed point of f. Q.E.D.

Since Corollary 1 can also be proved by defining first $A = \{x \in J\colon x \ge f(x)\}$, and then proving $x^* = \sup A \in A$, analogous to the proof of Theorem 2 we now have

COROLLARY 2. *Let* $f: K_n \to \mathbf{R}^n$ *be such that* (3.2) *is satisfied,* $-f$ *is quasi-monotone and*

$$f_i(x - x_i e_i) \leq 0 \quad \text{and} \quad 1 \leq f_i(x + (1 - x_i)e_i) \quad \text{on } K_n$$

for all $1 \leq i \leq n$. *Then* f *has maximal and minimal fixed points in* K_n.

REMARK. It remains open whether the results in this paper can be extended to the case $X = l^\infty$ or l^p for some $p \geq 1$.

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