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SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

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ABSTRACT. Sufficient conditions are given for a nonlinear system of differential equations with deviating arguments to have solutions which approach finite limits as $t \to \infty$. No specific assumptions other than continuity are imposed on the deviating arguments. The nonlinearities may be superlinear, sublinear, or singular in form, or a mixture of these. Some of the results are global.

We consider the system of functional differential equations with deviating arguments

$$x_i'(t) = f_i(t, x_1(g_{i1}(t)), \dots, x_n(g_{in}(t))), \qquad 1 \le i \le n, \tag{1}$$

where $f_i\colon [a,\infty) \times \mathbf{R}^n \to \mathbf{R}$ and $g_{ij}\colon [a,\infty) \to \mathbf{R}$, $1 \le i,j \le n$, are continuous. An important special case of (1) is

$$x_i'(t) = \sum_{j=1}^{n} a_{ij}(t)|x_j(g_{ij}(t))|^{\gamma_{ij}} \operatorname{sgn}[x_j(g_{ij}(t))], \qquad 1 \le i \le n, \tag{2}$$

where $a_{ij}\colon [a,\infty) \to \mathbf{R}$ are continuous and γ_{ij} are nonzero real constants, $1 \le i,j \le n$. For notational convenience, we abbreviate (1) as

$$x_i'(t) = f_i(t; X), \qquad 1 \le i \le n,$$

or, in terms of vector notation, as

$$X'(t) = F(t; X).$$

There has been considerable recent interest in the study of the oscillatory behavior of solutions of differential systems of the form (1); see, e.g. [**1–7**]. To the best of the authors' knowledge, however, a systematic existence theory for solutions of (1) has not yet been fully developed. Our purpose here is to provide conditions guaranteeing the existence of solutions of (1) which are asymptotically constant at infinity.

We make no assumptions (other than continuity) on the deviating arguments $\{g_{ij}(t)\}$. For convenience in the case where some or all of these may be retarded for some values of t (perhaps even $\liminf_{t\to\infty} g_{ij}(t) = -\infty$ for some i,j), we make the following definition.

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DEFINITION 1. If $-\infty < t_0 < \infty$, then $\mathscr{C}_n(t_0)$ is the space of continuous n-vector functions $X = (x_1, \ldots, x_n)$ on $(-\infty, \infty)$ which are constant on $(-\infty, t_0]$, with the topology induced by the following definition of convergence:

$$X_j \to X \quad \text{as } j \to \infty$$

if

$$\lim_{j\to\infty}\left[\sup_{-\infty<t\le T}\|X_j(t)-X(t)\|\right]=0$$

for every T in $(-\infty, \infty)$. (Here $\|\cdot\|$ is any convenient vector norm.) Thus, $\mathscr{C}_n(t_0)$ is a Fréchet space.

In what follows we suppose that there exist continuous functions $f_i^*: [a,\infty) \times \mathbf{R}_+^n \to \mathbf{R}_+$, $1 \le i \le n$, such that

$$|f_i(t,\xi_1,\ldots,\xi_n)| \le f_i^*(t,|\xi_1|,\ldots,|\xi_n|), \qquad 1\le i\le n, \tag{3}$$

for all $(t, \xi_1, \ldots, \xi_n) \in [a, \infty) \times \mathbf{R}^n$.

THEOREM 1. *Suppose that $f_i^*(t, u_1, \ldots, u_n)$, $1 \le i \le n$, in (3) are nondecreasing in each u_j, $1 \le j \le n$, and that there are positive constants $\rho_1, \ldots, \rho_n, \theta$ and $t_0 \ge a$ such that*

$$\int_{t_0}^{\infty} f_i^*(t,\rho_1,\ldots,\rho_n)\,dt \le \frac{\theta}{1+\theta}\rho_i, \qquad 1\le i\le n. \tag{4}$$

Let $c_1, \ldots, c_n$ be constants such that

$$|c_i| \le \frac{\rho_i}{1+\theta}, \qquad 1\le i\le n. \tag{5}$$

Then there exists a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ such that

$$|\hat{x}_i(t)-c_i| \le \frac{\theta}{1+\theta}\rho_i, \qquad -\infty<t<\infty,\ 1\le i\le n, \tag{6}$$

$$\hat{X}'(t) = F(t;\hat{X}), \qquad t\ge t_0, \tag{7}$$

and

$$\lim_{t\to\infty}\hat{x}_i(t) = c_i, \qquad 1\le i\le n. \tag{8}$$

PROOF. Let S be the closed convex subset of $\mathscr{C}_n(t_0)$ consisting of functions X such that

$$|x_i(t)-c_i| \le \frac{\theta}{1+\theta}\rho_i, \qquad -\infty<t<\infty,\ 1\le i\le n. \tag{9}$$

Note that (5) and (9) imply the inequalities

$$|x_i(t)| \le \rho_i, \qquad -\infty<t<\infty,\ 1\le i\le n. \tag{10}$$

For $X \in S$, define $Y = TX$ by

$$y_i(t) = \begin{cases} c_i - \int_t^\infty f_i(s;X)\,ds, & t\ge t_0,\\ c_i - \int_{t_0}^\infty f_i(s;X)\,ds, & t<t_0, \end{cases} \qquad 1\le i\le n. \tag{11}$$

Because of (4) and (10), $\mathscr{T}(S) \subset S$. Lebesgue's dominated convergence theorem implies that if $\{Y_j\}$ is a sequence in S and $Y_j \to Y$ as $j \to \infty$, then $\{\mathscr{T} Y_j\}$

converges to $\mathscr{T}Y$ uniformly on $(-\infty, \infty)$, which is more than sufficient to show that $\mathscr{T}$ is continuous. Differentiating (11) shows that

$$(\mathscr{T}X)'(t) = \begin{cases} F(t;X), & t \geq t_0^+, \\ 0, & t \leq t_0^-, \end{cases}$$

with appropriate one-sided interpretations at t_0. Now the Arzela theorem implies that $\mathscr{T}(S)$ has compact closure. Hence, $\mathscr{T}$ satisfies the hypotheses of the Schauder-Tychonoff fixed point theorem on S, and so there exists an $\hat{X}$ in S such that $\hat{X} = \mathscr{T}\hat{X}$. It is easy to see that $\hat{X}$ satisfies (6), (7) and (8). This completes the proof.

The following theorem is "local" near infinity.

THEOREM 2. *Suppose that $f_i^*(t, u_1, \ldots, u_n)$, $1 \leq i \leq n$, in* (3) *are nondecreasing in each u_j, $1 \leq j \leq n$, and that*

$$\int_a^\infty f_i^*(t, \rho, \ldots, \rho)\, dt < \infty \tag{12}$$

for every $\rho > 0$. Let $c_1, \ldots, c_n$ be given constants and $\varepsilon > 0$. Then there exists a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ such that

$$|\hat{x}_i(t) - c_i| < \varepsilon, \qquad -\infty < t < \infty,\ 1 \leq i \leq n,$$

and (7) *and* (8) *hold, providing that t_0 is sufficiently large.*

PROOF. Choose $\rho > 0$ such that

$$|c_i| < \rho - \varepsilon, \qquad 1 \leq i \leq n,$$

and t_0 so that

$$\int_{t_0}^\infty f_i^*(t, \rho, \ldots, \rho)\, dt < \varepsilon, \qquad 1 \leq i \leq n.$$

Then apply Theorem 1 with $\rho_1 = \cdots = \rho_n = \rho$ and $\theta = \varepsilon/(\rho - \varepsilon)$.

DEFINITION 2. Suppose that the functions $f_i^*(t, u_1, \ldots, u_n)$ in (3) are all nondecreasing in $u_1, \ldots, u_n$ and the functions $u^{-1} f_i^*(t, u, \ldots, u)$ are monotonic in u for each $t \geq a$. Then we say that (1) is *purely superlinear* if

$$\lim_{u \to 0+} u^{-1} f_i^*(t, u, \ldots, u) = 0, \qquad t \geq a,\ 1 \leq i \leq n, \tag{13}$$

or *purely sublinear* if

$$\lim_{u \to \infty} u^{-1} f_i^*(t, u, \ldots, u) = 0, \qquad t \geq a,\ 1 \leq i \leq n. \tag{14}$$

According to this defintion, (2) is purely superlinear if $\gamma_{ij} > 1$ for $1 \leq i, j \leq n$, or purely sublinear if $0 < \gamma_{ij} < 1$ for $1 \leq i, j \leq n$.

We now consider global existence theorems which guarantee the existence of solutions of (1) on a *given* interval $[t_0, \infty)$.

THEOREM 3. *Suppose that* (1) *is purely superlinear and that* (12) *holds for some $\rho > 0$. Let $t_0 \geq a$ and $\theta > 0$ be given. Then there is a number $\rho_0 > 0$ such that if*

$$|c_i| \leq \frac{\rho_0}{1 + \theta}, \qquad 1 \leq i \leq n, \tag{15}$$

then there is a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ which satisfies (7), (8), *and the inequalities*

$$|\hat{x}_i(t) - c_i| \le \frac{\theta}{1+\theta}\rho_0, \qquad -\infty < t < \infty, \ 1 \le i \le n.$$

PROOF. From (13) and the Lebesgue dominated convergence theorem, there is a sufficiently small constant $\rho_0 > 0$ such that

$$\int_{t_0}^{\infty} f_i^*(t, \rho_0, \dots, \rho_0)\, dt \le \frac{\theta}{1+\theta}\rho_0. \tag{16}$$

Now apply Theorem 1 with $\rho_1 = \cdots = \rho_n = \rho_0$ to obtain the stated conclusion.

THEOREM 4. *Let* (1) *be purely sublinear and suppose that* (12) *holds for some $\rho > 0$. Suppose that $t_0 \ge a$ and $c_1, \dots, c_n$ are arbitrary constants. Then there is a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ which satisfies* (7) *and* (8).

PROOF. Let $\theta > 0$ be arbitrary. Then choose $\rho_0 > 0$ so large that (15) and (16) both hold. (We can satisfy (16) with large ρ_0 because of (14).) Now apply Theorem 1 with $\rho_1 = \cdots = \rho_n = \rho_0$.

DEFINITION 3. Suppose that in (1) the functions $f_i\colon [a,\infty) \times [\mathbf{R}\backslash\{0\}]^n \to \mathbf{R}$, $1 \le i \le n$, are continuous. Then the system (1) is said to be *purely singular* if there exist continuous functions $f_i^*(t, u_1, \dots, u_n)$, $1 \le i \le n$, on $[a,\infty) \times (0,\infty)^n$ which are nonincreasing in each u_j, $1 \le j \le n$, and satisfy (3) for all $(t, \xi_1, \dots, \xi_n) \in [a,\infty) \times [\mathbf{R}\backslash\{0\}]^n$.

The system (2) is purely singular if $\gamma_{ij} < 0$ for all $1 \le i, j \le n$.

THEOREM 5. *Suppose that* (1) *is purely singular and that*

$$\int_a^{\infty} f_i^*(t, \rho_1, \dots, \rho_n)\, dt < \infty, \qquad 1 \le i \le n,$$

for some positive numbers $\rho_1, \dots, \rho_n$. Let $t_0 \ge a$ be given. Then there is a number θ in $(0,1)$ with the following property: if $c_1, \dots, c_n$ are any constants such that

$$|c_i| \ge \frac{\rho_i}{1-\theta}, \qquad 1 \le i \le n, \tag{17}$$

then there exists a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ which satisfies (7), (8) *and the inequalities*

$$|x_i(t) - c_i| \le \frac{\theta}{1-\theta}\rho_i, \qquad -\infty < t < \infty, \ 1 \le i \le n.$$

PROOF. In this case let S be the set of functions X in $\mathscr{C}_n(t_0)$ such that

$$|x_i(t) - c_i| \le \frac{\theta}{1-\theta}\rho_i, \qquad -\infty < t < \infty, \ 1 \le i \le n. \tag{18}$$

It is easy to show that (17) and (18) imply the inequalities

$$|x_i(t)| \ge \rho_i, \qquad -\infty < t < \infty, \quad 1 \le i \le n. \tag{19}$$

Let $\mathscr{T}$ be as in the proof of Theorem 1 (cf. (11)). Because of (19) and the pure singularity of (1), it is straightforward to verify that if S is as in (18), then $\mathscr{T}(S) \subset S$ if

$$\int_{t_0}^{\infty} f_i^*(t, \rho_1, \dots, \rho_n)\, dt \le \frac{\theta}{1-\theta}\rho_i, \qquad 1 \le i \le n.$$

These inequalities can be forced to hold by choosing θ sufficiently close to 1. The rest of the proof is the same as that of Theorem 1.

When specialized to the system (2), the above theorems yield the following results.

COROLLARY 1. *Suppose that*

$$\int_a^\infty |a_{ij}(t)|\,dt < \infty, \qquad 1 \le i,j \le n. \tag{20}$$

(i) *If* $\gamma_{ij} > 0$, $1 \le i,j \le n$, *then there exists a solution* $\hat{X}$ *of* (2) *which is defined in some neighborhood of* ∞ *and satisfies* (8) *for any given constants* $c_1, \ldots, c_n$.

(ii) *If* $0 < \gamma_{ij} < 1$, $1 \le i,j \le n$, *then there exists a solution* $\hat{X}$ *of* (2) *which is defined on a given interval* $[t_0, \infty)$ *and satisfies* (8) *for any given constants* $c_1, \ldots, c_n$.

(iii) *If* $\gamma_{ij} > 1$, $1 \le i,j \le n$, *then there exists a solution* $\hat{X}$ *of* (2) *which is defined on a given interval* $[t_0, \infty)$ *and satisfies* (8), *provided that* $|c_1|, \ldots, |c_n|$ *are sufficiently small.*

(iv) *If* $\gamma_{ij} < 0$, $1 \le i,j \le n$, *then there exists a solution* $\hat{X}$ *of* (2) *which is defined on a given interval* $[t_0, \infty)$ *and satisfies* (8), *provided that* $|c_1|, \ldots, |c_n|$ *are sufficiently large.*

It is also possible to obtain some results if (1) is singular with respect to some variables and nonsingular with respect to others. We illustrate this for the system (2).

THEOREM 6. *Suppose that A and B are nonempty subsets of* $\{1, 2, \ldots, n\}$ *such that* $A \cap B = \emptyset$ *and* $A \cup B = \{1, 2, \ldots, n\}$. *Let*

$$\gamma_{ij} > 0, \qquad 1 \le i \le n, \ \text{if } j \in A \tag{21}$$

(*i.e.*, (2) *is nonsingular with respect to* x_i *for* $i \in A$) *and*

$$\gamma_{ij} < 0, \qquad 1 \le i \le n, \ \text{if } j \in B \tag{22}$$

(*i.e.*, (2) *is singular with respect to* x_i *for* $i \in B$). *Suppose that* (20) *holds and that there are constants* ρ_i ($i \in A$) *and* $t_0 \ge a$ *such that*

$$\sum_{j \in A} \rho_j^{\gamma_{ij}} \int_{t_0}^\infty |a_{ij}(t)|\,dt < \rho_i, \qquad i \in A. \tag{23}$$

Let $c_1, \ldots, c_n$ *be given constants. Then there exists a function* $\hat{X}$ *in* $\mathscr{C}_n(t_0)$ *which satisfies* (7) *and* (8), *provided that* $|c_i|$ *is sufficiently small for i in A, or sufficiently large for i in B.*

PROOF. Choose positive constants r_j ($j \in B$) and α sufficiently large so that

$$\sum_{j \in A} \rho_j^{\gamma_{ij}} \int_{t_0}^\infty |a_{ij}(t)|\,dt + \sum_{j \in B} r_j^{\gamma_{ij}} \int_{t_0}^\infty |a_{ij}(t)|\,dt \le \frac{\alpha}{1+\alpha}\rho_i, \qquad i \in A. \tag{24}$$

(This is possible because of (22) and (23).) Now pick β in $(0,1)$ so that

$$\sum_{j \in A} \rho_j^{\gamma_{ij}} \int_{t_0}^\infty |a_{ij}(t)|\,dt + \sum_{j \in B} r_j^{\gamma_{ij}} \int_{t_0}^\infty |a_{ij}(t)|\,dt \le \frac{\beta}{1-\beta} r_i, \qquad i \in B. \tag{25}$$

Let

$$|c_i| \le \frac{\rho_i}{1+\alpha}, \qquad i \in A, \tag{26}$$

and

$$|c_i| \ge \frac{r_i}{1-\beta}, \qquad i \in B. \tag{27}$$

Let S be the subset of $\mathscr{C}_n(t_0)$ such that

$$|x_i(t) - c_i| \le \frac{\alpha}{1+\alpha}\rho_i, \qquad -\infty < t < \infty,\ i \in A, \tag{28}$$

and

$$|x_i(t) - c_i| \le \frac{\beta}{1-\beta}r_i, \qquad -\infty < t < \infty,\ i \in B. \tag{29}$$

Define the mapping $\mathscr{T} : S \to \mathscr{C}_n(t_0)$ by (11) with

$$f_i(t;X) = \sum_{j=1}^{n} a_{ij}(t)|x_j(g_{ij}(t))|^{\gamma_{ij}} \operatorname{sgn}[x_j(g_{ij}(t))], \qquad 1 \le i \le n.$$

Since (26) and (28) imply that

$$|x_i(t)| \le \rho_i, \qquad -\infty < t < \infty,\ i \in A,$$

while (27) and (29) imply that

$$|x_i(t)| \ge r_i, \qquad -\infty < t < \infty,\ i \in B,$$

(21), (22), (24) and (25) imply that $\mathscr{T}(S) \subset S$. The rest of the proof is like that of Theorem 1.

If the nonsingular part of (2) is purely sublinear or purely superlinear, then it is always possible to satisfy (23) with any $t_0 \ge a$ by choosing ρ_i $(i \in A)$ appropriately. This implies the following global result.

COROLLARY 2. *Let A and B be a partition of $\{1, 2, \ldots, n\}$, as in Theorem* 6, *and suppose that* (20) *and* (22) *hold. Let t_0 be given, $t_0 \ge a$.*

(i) *Suppose that $0 < \gamma_{ij} < 1$, $1 \le i \le n$, if $j \in A$. Let c_i $(i \in A)$ be arbitrary. Then there is a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ which satisfies* (7) *and* (8), *provided that $|c_i|$ $(i \in B)$ are sufficiently large.*

(ii) *Suppose that $\gamma_{ij} > 1$, $1 \le i \le n$, if $j \in A$. Then there is a function $\hat{X}$ in $\mathscr{C}_n(t_0)$ which satisfies* (7) *and* (8), *provided that $|c_i|$ $(i \in A)$ are sufficiently small and $|c_i|$ $(i \in B)$ are sufficiently large.*

REFERENCES

1. I. Foltynska and J. Werbowski, *On the oscillatory behavior of solutions of systems of differential equations with deviating arguments*, Colloq. Math. Soc. János Bolyai 30: *Qualitative Theory of Differential Equations* (M. Farkas, ed.), Vol. 1, Szeged, 1981, pp. 243–256.
2. Y. Kitamura and T. Kusano, *Oscillation and a class of nonlinear differential systems with general deviating arguments*, Nonlinear Anal. **2** (1978), 537–551.
3. P. Marušiak, *Oscillatory properties of solutions of nonlinear differential systems with deviating arguments*, Czechoslovak Math. J. **36** (1986), 223–231.
4. V. Šeda, *On nonlinear differential systems with deviating arguments*, Czechoslovak Math. J. **36** (1986), 450–466.
5. V. N. Shevelo, *Oscillation of solutions of differential equations with deviating argument*, Naukova Dumka, Kiev, 1978. (Russian)

6. V. N. Shevelo, N. V. Varch and A. G. Gritsai, *Oscillatory properties of solutions of systems of differential equations with deviating argument*, Preprint 85.10, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1985. (Russian)

7. W. F. Trench, *Asymptotics of differential systems with deviating arguments*, Proc. Amer. Math. Soc. **92** (1984), 219–224.

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