ON STABILITY OF ENDMORPHISMS

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ABSTRACT. In this note we prove a generalization of R. Mañé's theorem. R. Mañé proved that $C^r$ absolutely stable endomorphisms satisfy Axiom A. We prove that if an endomorphism $f$ is both $C^r$ structurally and infinitesimally stable, then $f$ satisfies Axiom A.

1. Introduction. This note is a generalization of R. Mañé's theorem on absolute stability [4]. We prove this theorem under weaker hypotheses. Our Theorem is the following

THEOREM. If $f \in \text{End}^r(M)$ is both $C^r$ structurally and infinitesimally stable, then $f$ satisfies Axiom A.

First we establish some background. Let $M$ be a compact connected smooth manifold without boundary and let $\text{End}^r(M)$, $r \geq 1$, be the space of $C^r$ endomorphisms of $M$ endowed with the $C^r$ topology. We say that $f$ is $C^r$ structurally stable if there exists a neighborhood $\mathcal{U}$ of $f$ in $\text{End}^r(M)$ such that for every $g \in \mathcal{U}$ there exists a homeomorphism $h$ of $M$ satisfying $fh = hg$. If $f \in \text{End}^r(M)$, let $\text{Per}(f)$ be the set of all periodic points of $f$ and let $\Omega(f) = \{x \in M\}$ for every neighborhood $U$ of $x$, there exists $n > 0$ with $f^n(U) \cap U \neq \emptyset$. We denote by $S(f)$ the set of singularities of $f$, i.e. those points $x$ of $M$ where $Tf|T_xM$ is not injective.

We give two definitions following R. Mañé [4].

DEFINITION. We say that $f \in \text{End}^r(M)$ is $C^r$ absolutely stable if there exist a neighborhood $\mathcal{U}$ of $f$ in $\text{End}^r(M)$ and a constant $K > 0$ such that for all $g \in \mathcal{U}$ there exists a homeomorphism $h$ of $M$ satisfying $gh = hf$ and $d(h, I) \leq Kd(f, g)$ where $d(\cdot, \cdot)$ is defined by $d(f_1, f_2) = \sup\{|\rho(f_1(x), f_2(x))| x \in M\}$, $\rho(\cdot, \cdot)$ being a metric on $M$ and $I$ is the identity map of $M$.

DEFINITION. We say that $f \in \text{End}^r(M)$ satisfies Axiom A if there exist a continuous splitting $TM|\Omega(f) = E^s \oplus E^u$, and a Riemannian metric $|\cdot|$ on $M$, and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)E^s \subset E^s$, $(Tf)E^u = E^u$;
(b) $|(Tf)^nE^s_x| \leq K\lambda^n$ for $x \in \Omega(f)$, $n > 0$,
(c) $|(Tf)^nE^u_x| \geq K\lambda^{-n}|v|$ for $x \in \Omega(f)$, $v \in E^u_x$, $n > 0$;
(d) if $x_1 \neq x_2 \in \Omega(f)$ and $f(x_1) = f(x_2) = x$, then $E^s_x = \{0\}$;
(e) $S(f) \cap \Omega(f) = \emptyset$.

In [8] the following conjecture was stated.

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CONJECTURE. If \( f \in \text{End}^r(M) \) is \( C^r \) structurally stable, then \( \Omega(f) \cap S(f) = \emptyset \). In fact, \( \text{Per}(f) \cap S(f) = \emptyset \) holds if \( f \) is \( C^r \) structurally stable, and \( \Omega(f) \cap S(f) = \emptyset \) holds if \( f \) is \( C^r \) absolutely stable. The latter property is a part of the theorem by R. Mañé [4].

**Theorem M.** \( C^r \) absolutely stable endomorphisms satisfy Axiom A.

On the other hand, F. Przytycki defined Axiom A somewhat different from Mañé’s Axiom A, and found sufficient conditions for \( \Omega \)-stability of an endomorphism [6, 7].

Our Theorem is a generalization of Theorem M.

2. Proof of the Theorem. In this section we will prove our Theorem. To prove the Theorem we need some definitions and lemmas. If \( \Lambda \subset M \) is a compact subset of \( M \) let \( \Gamma^0(\Lambda) \) be the space of bounded sections of \( TM|\Lambda \) with the norm \( ||\eta|| = \sup\{||\eta(x)|| | x \in \Lambda\} \) and let \( \Gamma^0(\Lambda) \) be the closed subspace of continuous sections. If \( f \in \text{End}^r(M) \) and \( \Gamma(\Lambda) \subset \Lambda \) let \( T_f M|\Lambda \) be the vector bundle on \( \Lambda \) consisting of couples \( (p,v) \) with \( p \in \Lambda, v \in T_f(p)M \). Let \( \Gamma_f(\Lambda), \Gamma_f^0(\Lambda) \) be the corresponding spaces of bounded and continuous sections of \( T_f M|\Lambda \).

We define the linear operator \( L_f: \Gamma^0(\Lambda) \to \Gamma_f^0(\Lambda) \) by

\[
L_f(\eta) = (T_f) \circ \eta - \eta \circ f \quad \text{for } \eta \in \Gamma^0(\Lambda).
\]

**Definition.** Let \( f \in \text{End}^r(M) \) and let \( \Lambda \subset M \) be a compact subset with \( f(\Lambda) = \Lambda \). We say that \( \Lambda \) is a prehyperbolic set for \( f \) if there exist a continuous splitting \( TM|\Lambda = E^s \oplus E^u \), and a Riemannian metric \( || \cdot || \) on \( M \) and constants \( K > 0, 0 < \lambda < 1 \) satisfying:

(a) \( (T_f)^n E^s \subset E^s \), \( (T_f)^n E^u = E^u \);

(b) \[ |(T_f)^n E^s_x| \leq K \lambda^n |x| \] for \( x \in \Lambda, n > 0 \); \[ |(T_f)^n v| \geq K \lambda^{-n} |v| \] for \( x \in \Lambda, v \in E^u_x, n > 0 \);

(c) if \( x_1 \neq x_2 \in \Lambda \) and \( f(x_1) = f(x_2) = x \), then \( E^s_x = \{0\} \).

We say that \( f \in \text{End}^r(M) \) is infinitesimally stable if the linear operator \( L_f: \Gamma^0(M) \to \Gamma_f^0(M) \) is surjective. It is evident that the absolute stability of \( f \) implies the structural stability of \( f \). Also it is easy to prove that if \( f \in \text{End}^r(M) \) is absolutely stable, then \( f \) is infinitesimally stable by the similar argument for the case of diffeomorphisms in [1]. If \( x \in M \) let \( \omega(x) = \omega(x,f) \) be the set of \( \omega \)-limit points of \( x \) for \( f \) and let \( \overline{L}^+(f) \) be the closure of \( L^+(f) = \{\omega(x)|x \in M\} \).

In the proof of the theorem, we shall use the following two lemmas of [4].

**Lemma 1.** If \( f \in \text{End}^r(M) \) is infinitesimally stable, then \( \overline{L}^+(f) \) is prehyperbolic.

**Lemma 2.** Suppose that \( f \in \text{End}^r(M) \) and \( \overline{L}^+(f) \) is prehyperbolic. Then given \( x \in \overline{L}^+(f) \) and a neighborhood \( U \) of \( f \) there exist a neighborhood \( U \) of \( x \) and \( g \in U \) such that:

(a) \( g(y) = f(y) \) for all \( y \in U \);

(b) \( x \in \text{Per}(g) \).

From the argument of R. Mañé [4], it clearly follows that we only need to prove the following proposition in order to prove the Theorem.
Proposition. If \( f \in \text{End}^r(M) \) is both \( C^r \) structurally and infinitesimally stable, then \( \overline{L}^+(f) = \overline{\text{Per}}(f) \).

To prove the Proposition, we introduce the concept of \( P \)-stability. \( P = \overline{\text{Per}}(f) \) will denote the closure of the set of periodic points of \( f \). We say that \( f \in \text{End}^r(M) \) is \( P \)-stable if there exists a neighborhood \( \mathcal{U} \) of \( f \) in \( \text{End}^r(M) \) such that for each \( g \in \mathcal{U} \) there exists a homeomorphism \( h : \overline{\text{Per}}(f) \to \overline{\text{Per}}(g) \) satisfying \( hf = gh \) [5]. It is obvious that the structural stability implies the \( P \)-stability.

Proof of Proposition. We first suppose that \( \overline{\text{Per}}(f) \subseteq \overline{L}^+(f) \). We show that \( \overline{L}^+(f) \cap S(f) = \emptyset \). Suppose that \( x \in \overline{L}^+(f) \cap S(f) \). By Lemmas 1 and 2, we can take \( g \in \text{End}^r(M) \) nearby \( f \) coinciding with \( f \) in a neighborhood of \( x \) (therefore \( x \in S(g) \)) and such that \( x \in \text{Per}(g) \). Hence \( x \in \text{Per}(g) \cap S(g) \). This contradicts the stability of \( g \). Hence \( \overline{L}^+(f) \cap S(f) = \emptyset \). Also it is obvious that \( \overline{\text{Per}}(f) \) is prehyperbolic. Therefore \( \overline{\text{Per}}(f) \) has a decomposition, \( \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n \) into disjoint prehyperbolic sets, following the method of Przytycki [7] and Newhouse [5]. Here each \( \Lambda_i \) is a prehyperbolic set such that the periodic points are dense in \( \Lambda_i \). Let \( U_i \) be a compact neighborhood of \( \Lambda_i \) for \( 1 \leq i \leq n \). If we choose \( U_i \) small enough, we have that \( U_i \cap U_j = \emptyset \) if \( i \neq j \) and \( \overline{L}^+(f) \setminus \bigcup_{1 \leq i \leq n} U_i \neq \emptyset \).

Now suppose that \( f|\Lambda_i \) is injective. Then shrinking \( U_i \) if necessary, \( f|U_i \) is a diffeomorphism onto its image. By Theorem 7.3 of [2], there is a neighborhood \( \mathcal{N}_i \) of \( f \) in \( \text{End}^r(M) \) such that \( g \in \mathcal{N}_i \) then there exists a homeomorphism \( h \) close to identity which conjugates \( \Lambda_i \) with a \( g \)-invariant subset of \( U_i \).

Next suppose that \( f|\Lambda_i \) is not injective. Then it is easy to show that \( f|\Lambda_i \) is a (quasi-) expanding map, i.e. \( \dim E^u_x = \dim M \) for any \( x \in \Lambda_i \). By the similar argument to Przytycki [7], there exists a neighborhood \( \mathcal{N}_i \) of \( f \) in \( \text{End}^r(M) \) such that if \( g \in \mathcal{N}_i \) then there exists a homeomorphism \( h \) from \( \Lambda_i \) onto its image satisfying \( gh = hf \) and \( h(\Lambda_i) \subseteq U_i \).

From above arguments and \( P \)-stability of \( f \), it follows that there exists a neighborhood \( \mathcal{N} \) of \( f \) in \( \text{End}^r(M) \) such that \( \overline{\text{Per}}(g) \subseteq \bigcup_{1 \leq i \leq n} U_i \) for any \( g \in \mathcal{N} \). Let \( x \in \overline{L}^+(f) \setminus \bigcup_{1 \leq i \leq n} U_i \). By Lemma 2, there exists \( g \in \mathcal{N} \) such that \( x \) is a periodic point of \( g \). This is a contradiction.

Subsequently we obtain the Corollary of the Theorem:

Corollary. If \( f \in \text{End}^r(M) \) is both \( C^r \) structurally and infinitesimally stable, then \( f \) satisfies Axiom A and no cycle condition.

References


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