ON STABILITY OF ENDOmorphisms

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ABSTRACT. In this note we prove a generalization of R. Mañé's theorem. R. Mañé proved that C\^r absolutely stable endomorphisms satisfy Axiom A. We prove that if an endomorphism f is both C\^r structurally and infinitesimally stable, then f satisfies Axiom A.

1. Introduction. This note is a generalization of R. Mañé's theorem on absolute stability [4]. We prove this theorem under weaker hypotheses. Our Theorem is the following

THEOREM. If f \in End^r(M) is both C\^r structurally and infinitesimally stable, then f satisfies Axiom A.

First we establish some background. Let M be a compact connected smooth manifold without boundary and let End^r(M), r \geq 1, be the space of C\^r endomorphisms of M endowed with the C\^r topology. We say that f is C\^r structurally stable if there exists a neighborhood U of f in End^r(M) such that for every g \in U there exists a homeomorphism h of M satisfying fh = hg. If f \in End^r(M), let Per(f) be the set of all periodic points of f and let \Omega(f) = \{x \in M\} for every neighborhood U of x, there exists n > 0 with f^n(U) \cap U \neq \emptyset. We denote by S(f) the set of singularities of f, i.e. those points x of M where T|TF_xM is not injective.

We give two definitions following R. Mañé [4].

DEFINITION. We say that f \in End^r(M) is C\^r absolutely stable if there exist a neighborhood U of f in End^r(M) and a constant K > 0 such that for all g \in U there exists a homeomorphism h of M satisfying gh = hf and d(h,I) \leq Kd(f,g) where d(\cdot,\cdot) is defined by d(f_1, f_2) = \sup\{\rho(f_1(x), f_2(x)) | x \in M\}, \rho(\cdot,\cdot) being a metric on M and I is the identity map of M.

DEFINITION. We say that f \in End^r(M) satisfies Axiom A if there exist a continuous splitting T|TM = Es \oplus Eu, and a Riemannian metric \| \cdot \| on M, and constants K > 0, 0 < \lambda < 1 satisfying:

(a) \langle Tf \rangle E^s \subset E^s, \langle Tf \rangle E^u = E^u;

(b) ||(Tf)^nE^s_x|| \leq K\lambda^n \text{ for } x \in \Omega(f), n > 0,

(c) \langle Tf \rangle^nv \geq K\lambda^{-n}|v| \text{ for } x \in \Omega(f), v \in E^u_x, n > 0;

(d) if x_1 \neq x_2 \in \Omega(f) and f(x_1) = f(x_2) = x, then E^s_x = \{0\};

(e) S(f) \cap \Omega(f) = \emptyset.

In [8] the following conjecture was stated.

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CONJECTURE. If $f \in \text{End}^r(M)$ is $C^r$ structurally stable, then $\Omega(f) \cap S(f) = \emptyset$. In fact, $\text{Per}(f) \cap S(f) = \emptyset$ holds if $f$ is $C^r$ structurally stable, and $\Omega(f) \cap S(f) = \emptyset$ holds if $f$ is $C^r$ absolutely stable. The latter property is a part of the theorem by R. Mañé [4].

THEOREM M. $C^r$ absolutely stable endomorphisms satisfy Axiom A.

On the other hand, F. Przytycki defined Axiom A somewhat different from Mañé's Axiom A, and found sufficient conditions for $\Omega$-stability of an endomorphism [6, 7].

Our Theorem is a generalization of Theorem M.

2. Proof of the Theorem. In this section we will prove our Theorem. To prove the Theorem we need some definitions and lemmas. If $\Lambda \subset M$ is a compact subset of $M$ let $\Gamma^0(\Lambda)$ be the space of bounded sections of $TM|\Lambda$ with the norm $||\eta|| = \sup \{||\eta(x)|| | x \in \Lambda\}$ and let $\Gamma^0(\Lambda)$ be the closed subspace of continuous sections. If $f \in \text{End}^r(M)$ and $f(\Lambda) \subset \Lambda$ let $T_fM|\Lambda$ be the vector bundle on $\Lambda$ consisting of couples $(p,v)$ with $p \in \Lambda$, $v \in T_f(p)M$. Let $\Gamma^0_f(\Lambda)$ be the corresponding spaces of bounded and continuous sections of $T_fM|\Lambda$.

We define the linear operator $L_f: \Gamma^0(\Lambda) \to \Gamma^0_f(\Lambda)$ by

$$L_f(\eta) = (Tf) \circ \eta - \eta \circ f \quad \text{for } \eta \in \Gamma^0(\Lambda).$$

**DEFINITION.** Let $f \in \text{End}^r(M)$ and let $\Lambda \subset M$ be a compact subset with $f(\Lambda) = \Lambda$. We say that $\Lambda$ is a [pre]hyperbolic set for $f$ if there exist a continuous splitting $TM|\Lambda = E^s \oplus E^u$, and a Riemannian metric $\|\|$ on $M$ and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)^nE^s \subset E^s$, $(Tf)^nE^u = E^u$;

(b) $|(Tf)^nE^s_x| \leq K\lambda^n$ for $x \in \Lambda$, $n > 0$,

$|(Tf)^nE^u_x| \geq K\lambda^{-n}|v|$ for $x \in \Lambda$, $v \in E^u_x$, $n > 0$;

(c) if $x_1 \neq x_2 \in \Lambda$ and $f(x_1) = f(x_2) = x$, then $E^s_x = \{0\}$.

We say that $f \in \text{End}^r(M)$ is [in]finitesimally stable if the linear operator $L_f: \Gamma^0(\Lambda) \to \Gamma^0_f(\Lambda)$ is surjective. It is evident that the absolute stability of $f$ implies the structural stability of $f$. Also it is easy to prove that if $f \in \text{End}^r(M)$ is absolutely stable, then $f$ is infinitesimally stable by the similar argument for the case of diffeomorphisms in [1]. For $x \in M$ let $\omega(x) = \omega(x,f)$ be the set of $\omega$-limit points of $x$ for $f$ and let $\bar{L}^+(f)$ be the closure of $L^+(f) = \{\omega(x) | x \in M\}$.

In the proof of the theorem, we shall use the following two lemmas of [4].

**LEMMA 1.** If $f \in \text{End}^r(M)$ is infinitesimally stable, then $\bar{L}^+(f)$ is prehyperbolic.

**LEMMA 2.** Suppose that $f \in \text{End}^r(M)$ and $\bar{L}^+(f)$ is prehyperbolic. Then given $x \in \bar{L}^+(f)$ and a neighborhood $U$ of $f$ there exist a neighborhood $U$ of $x$ and $g \in U$ such that:

(a) $g(y) = f(y)$ for all $y \in U$;

(b) $x \in \text{Per}(g)$.

From the argument of R. Mañé [4], it clearly follows that we only need to prove the following proposition in order to prove the Theorem.
PROPOSITION. If $f \in \text{End}^r(M)$ is both $C^r$ structurally and infinitesimally stable, then $L^+(f) = \text{Per}(f)$.

To prove the Proposition, we introduce the concept of $\overline{P}$-stability. $\overline{P} = \overline{\text{Per}(f)}$ will denote the closure of the set of periodic points of $f$. We say that $f \in \text{End}^r(M)$ is $\overline{P}$-stable if there exists a neighborhood $\mathcal{U}$ of $f$ in $\text{End}^r(M)$ such that for each $g \in \mathcal{U}$ there exists a homeomorphism $h : \text{Per}(f) \to \text{Per}(g)$ satisfying $hf = gh$ [5]. It is obvious that the structural stability implies the $\overline{P}$-stability.

**Proof of Proposition.** We first suppose that $\overline{\text{Per}(f)} \subsetneq L^+(f)$. We show that $L^+(f) \cap S(f) = \emptyset$. Suppose that $x \in L^+(f) \cap S(f)$. By Lemmas 1 and 2, we can take $g \in \text{End}^r(M)$ nearby $f$ coinciding with $f$ in a neighborhood of $x$ (therefore $x \in S(g)$) and such that $x \in \text{Per}(g)$. Hence $x \in \text{Per}(g) \cap S(g)$. This contradicts the stability of $g$. Hence $L^+(f) \cap S(f) = \emptyset$. Also it is obvious that $\text{Per}(f)$ is prehyperbolic. Therefore $\overline{\text{Per}(f)}$ has a decomposition, $\Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n$ into disjoint prehyperbolic sets, following the method of Przytycki [7] and Newhouse [5]. Here each $\Lambda_i$ is a prehyperbolic set such that the periodic points are dense in $\Lambda_i$. Let $U_i$ be a compact neighborhood of $\Lambda_i$ for $1 \leq i \leq n$. If we choose $U_i$ small enough, we have that $U_i \cap U_j = \emptyset$ if $i \neq j$ and $L^+(f) \setminus \bigcup_{1 \leq i \leq n} U_i \neq \emptyset$.

Now suppose that $f|\Lambda_i$ is injective. Then shrinking $U_i$ if necessary, $f|U_i$ is a diffeomorphism onto its image. By Theorem 7.3 of [2], there is a neighborhood $\mathcal{N}_i$ of $f$ in $\text{End}^r(M)$ such that if $g \in \mathcal{N}_i$ then there exists a homeomorphism $h$ close to identity which conjugates $\Lambda_i$ with a $g$-invariant subset of $U_i$.

Next suppose that $f|\Lambda_i$ is not injective. Then it is easy to show that $f|\Lambda_i$ is a (quasi-) expanding map, i.e. $\dim E^u_x = \dim M$ for any $x \in \Lambda_i$. By the similar argument to Przytycki [7], there exists a neighborhood $\mathcal{N}_i$ of $f$ in $\text{End}^r(M)$ such that if $g \in \mathcal{N}_i$ then there exists a homeomorphism $h$ from $\Lambda_i$ onto its image satisfying $gh = hf$ and $h(\Lambda_i) \subset U_i$.

From above arguments and $\overline{P}$-stability of $f$, it follows that there exists a neighborhood $\mathcal{N}$ of $f$ in $\text{End}^r(M)$ such that $\overline{\text{Per}(g)} \subset \bigcup_{1 \leq i \leq n} U_i$ for any $g \in \mathcal{N}$. Let $x \in L^+(f) \setminus \bigcup_{1 \leq i \leq n} U_i$. By Lemma 2, there exists $g \in \mathcal{N}$ such that $x$ is a periodic point of $g$. This is a contradiction.

Subsequently we obtain the Corollary of the Theorem:

**Corollary.** If $f \in \text{End}^r(M)$ is both $C^r$ structurally and infinitesimally stable, then $f$ satisfies Axiom A and no cycle condition.

**References**


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