PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 4, December 1988

A LIFTING THEOREM AND ANALYTIC OPERATOR ALGEBRAS

TAKAHIKO NAKAZI

(Communicated by Paul S. Muhly)

Dedicated to Professor Shozo Koshi on his sixtieth birthday

ABSTRACT. Let K be a complex Hilbert space and H a closed subspace. It is shown that if a 2×2 selfadjoint operator matrix $\mathscr{T}$ with positive diagonals on $K \oplus K$ is positive on $H \oplus H^\perp$, then there exists a 2×2 operator matrix $\tilde{\mathscr{T}}$ with the same diagonals such that $\tilde{\mathscr{T}}$ is positive on $K \oplus K$ and $\mathscr{T}$ is the restriction of $\tilde{\mathscr{T}}$ to $H \oplus H^\perp$. When $\mathscr{T}$ is in a von Neumann algebra, we consider the problems of finding $\mathscr{T}$ in the same algebra. This lifting theorem has applications to weighted norm inequalities for conjugation operators on analytic operator algebras.

1. Introduction. Let $\mathscr{L}(K)$ be the set of all bounded linear operators on a complex Hilbert space K and $\mathscr{B}$ a von Neumann algebra on K. Let H be the closed subspace of K and $\mathscr{A}$ a (perhaps nonselfadjoint) weakly closed subalgebra of $\mathscr{B}$ which has H as an invariant subspace. We study 2×2 operator matrices $\mathscr{T} = (T_{ij})$ on $K \oplus K$ where $T_{ij} \in \mathscr{B}$ $(i, j = 1, 2)$, $T_{11} \geq 0$, $T_{22} \geq 0$ and $T_{21}^* = T_{12}$. $[\mathscr{B}]$ denotes the set of such operator matrices $\mathscr{T}$. $[\mathscr{A}]_0$ denotes the subset of $[\mathscr{B}]$ such that $T_{12} \in \mathscr{A}$ and $T_{11} = T_{22} = 0$.

Let us denote

$$\mathscr{T}[f_1, f_2] = \sum_{i,j=1}^{2} (T_{ij} f_i, f_j).$$

If $\mathscr{T}$ satisfies $\mathscr{T}[f_1, f_2] \geq 0$ for all f_1 in H (resp. K) and f_2 in $H^\perp$ (resp. K), then $\mathscr{T}$ is said to be positive on $H \oplus H^\perp$ (resp. $K \oplus K$) where $H^\perp$ is the orthogonal complement of H in K. $\mathscr{T}$ is positive on $H \oplus H^\perp$ if and only if compression $\mathscr{P}\mathscr{T}$ of $\mathscr{T}$ to $H \oplus H^\perp$ is positive where $\mathscr{P}$ is the orthogonal projection of $K \oplus K$ onto $H \oplus H^\perp$.

When $\mathscr{T} \in [\mathscr{B}]$ and $\mathscr{T}$ is positive on $H \oplus H^\perp$, we wish to $\tilde{\mathscr{T}}$ in $\mathscr{T} + [\mathscr{A}]_0$ which is positive on $K \oplus K$. Of course then $\mathscr{T}[f_1, f_2] = \tilde{\mathscr{T}}[f_1, f_2]$ for all f_1 in H and f_2 in $H^\perp$ or equivalently $\mathscr{P}\mathscr{T} | H \oplus H^\perp = \mathscr{P}\tilde{\mathscr{T}} | H \oplus H^\perp$. This problem is related with a lifting theorem of Cotlar and Sadosky [**4**] for the disk algebra and a generalized lifting theorem of the author and Yamamoto [**10**] for a general uniform algebra. In the special case, they consider the problem above when $\mathscr{B}$ is commutative. In our main theorm in §2, we do not assume that $\mathscr{B}$ is commutative and we establish a

Received by the editors December 9, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A20, 47A15, 47D25.

Key words and phrases. Commutant lifting, Arveson distance formula, factorization, Helson-Szegö theorem.

This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

lifting theorem. We then exhibit some examples in §3 and obtains two operator theoretic type Helson-Szegö in §4 (cf. [**6**, **7**]).

2. General lifting theorem. Let $\mathscr{F}$ be the subset of lat$\mathscr{A}$, the lattice of all $\mathscr{A}$-invariant projections, then for any T in $\mathscr{B}$

$$\operatorname{dist}(T,\mathscr{A}) \geq \sup_{P\in\mathscr{F}} \|(1-P)TP\|.$$

Many concrete examples satisfy the following conditions (I) and (II) (see §3).

(I) There exists a subset $\mathscr{F}$ in lat$\mathscr{A}$ such that for any T in $\mathscr{B}$

$$\operatorname{dist}(T,\mathscr{A}) = \sup_{P\in\mathscr{F}} \|(1-P)TP\|.$$

(II) Every invertible positive operator in $\mathscr{B}$ has the form $A^*A = BB^*$, for some invertible A and B in $\mathscr{A}$.

(I) is a kind of Nehari's theorem for Hankel operators (cf. [**12**, **8**, **3**, **13** and **9**]). (II) is a factorization of Szegö-Cholesky type (cf. [**5**, **3** and **13**]). The following is a lifting theorem under the conditions (I) and (II). Then we say that $(\mathscr{B},\mathscr{A},\mathscr{F})$ has the lifting property when the conclusion of the theorem holds.

THEOREM. *Let $(\mathscr{B},\mathscr{A},\mathscr{F})$ satisfy the condition* (I) *and* (II). *If $\mathscr{T}$ in $[\mathscr{B}]$ is positive on $PK \oplus (1-P)K$ for every P in $\mathscr{F}$, then there exists $\tilde{\mathscr{T}}$ in $\mathscr{T} + [\mathscr{A}]_0$ which is positive on $K \oplus K$.*

PROOF. $\mathscr{T} = [T_{ij}]$ is positive on $PK \oplus (1-P)K$ if and only if for any $f \in PK$ and $g \in (1-P)K$

$$|(T_{12}f,g)|^2 \leq (T_{11}f,f)(T_{22}g,g).$$

By the hypothesis (II), for any positive integer n there exist two invertible operators A_n and B_n in $\mathscr{A}$ such that

$$T_{11} + 1/n = A_n^*A_n \quad \text{and} \quad T_{22} + 1/n = B_nB_n^*.$$

Since $P \in \operatorname{lat}\mathscr{A}$, $A_nPK = PK$ and $B_n^*(1-P)K = (1-P)K$, and hence for any $f \in PK$ and $g \in (1-P)K$

$$|(B_n^{-1}T_{12}A_n^{-1}f,g)|^2 \leq (f,f)(g,g).$$

By the hypothesis (I), for any positive number ε there exists $D_{n,\varepsilon}$ in $\mathscr{A}$ such that

$$\|B_n^{-1}T_{12}A_n^{-1} + D_{n,\varepsilon}\| \leq \sup \|(1-P)B_n^{-1}T_{12}A_n^{-1}P\| + \varepsilon \leq 1+\varepsilon.$$

Since $\mathscr{A}$ is weakly closed, there exists D_n in $\mathscr{A}$ such that

$$\|B_n^{-1}T_{12}A_n^{-1} + D_n\| \leq 1.$$

Put $S_n = B_nD_nA_n$, then S_n is in $\mathscr{A}$ and for any F, $G \in K$

$$|((T_{12}+S_n)F,G)|^2 \leq ((T_{11}+1/n)F,G)((T_{22}+1/n)F,G).$$

Again using the weakly closedness of $\mathscr{A}$, there exists S in $\mathscr{A}$ such that for any F, $G \in K$

$$|((T_{12}+S)F,G)|^2 \leq (T_{11}F,G)(T_{22}F,G)$$

and the theorem follows.

3. $(\mathscr{B}, \mathscr{A}, \mathscr{F})$ with the lifting property. The theorem shows that if $(\mathscr{B}, \mathscr{A}, \mathscr{F})$ satisfies (I) and (II) then it has a lifting property. Hence it is sufficient to check (I) and (II) in the following examples.

(1) Let $\mathscr{B} = \mathscr{L}(K)$, P an orthogonal projection and

$$\mathscr{A} = \{A \in \mathscr{L}(K)\colon PAP = AP\}.$$

It is clear that, for any $T \in \mathscr{L}(K)$, $\operatorname{dist}(T, \mathscr{A}) = \|(1-P)TP\|$. It is well known that if T is invertible then $T = A^*A = BB^*$ for some invertible A and B in $\mathscr{A}$ (see [**1**, Chapter I, **13**]). The theorem shows that $(\mathscr{L}(K), \mathscr{A}, P)$ has a lifting property.

(2) In (1) let U be a bilateral shift operator on K with $UP = PUP$. Suppose $\bigcap_{n=0}^{\infty} U^n(PK) = \{0\}$ and $\bigcup_{n=0}^{\infty} U^{*n}(PK)$ is dense in K. If $\mathscr{T} \in [\mathscr{L}(K)]$, $\mathscr{T} = [T_{ij}]$ and T_{ij} commutes with U for $i,j = 1,2$, then we can get a positive lifting operator $\tilde{\mathscr{T}} = [\tilde{T}_{ij}]$ in $\mathscr{T} + [\mathscr{A}]_0$ such that $\tilde{T}_{ij}$ commutes with U. This is a kind of commuting lifting (see [**14**]). In fact let $\mathscr{B}_1 = \{T \in \mathscr{L}(K)\colon UT = TU\}$, $\mathscr{A}_1 = \{A \in \mathscr{A}\colon UA = AU\}$ and $\mathscr{F} = \{P\}$, then (I) and (II) are true, too [**12** and **5**]. The theorem shows that $(\mathscr{B}_1, \mathscr{A}_1, P)$ has the lifting property.

(3) Let $\mathscr{B}$ be a factor with faithful semifinite normal trace τ and let $\mathscr{E}$ be a complete nest of selfadjoint projections in $\mathscr{B}$. We write $\operatorname{Alg}\mathscr{E}$ for the nest algebra associated with $\mathscr{E}$, and let

$$\mathscr{A} = \mathscr{B} \cap \operatorname{Alg}\mathscr{E}.$$

(I) with $\mathscr{F} = \mathscr{E}$ is a variant of Arveson's distance formula, that is,

$$\operatorname{dist}(T, \mathscr{A}) = \sup_{P \in \mathscr{F}} \|(1-P)TP\|$$

(see [**3**]). Assuming $\mathscr{E}$ is well-ordered, (II) is valid by [**13**]. The theorem shows that $(\mathscr{B}, \mathscr{A}, \mathscr{F})$ has the lifting property.

(4) In (3) let

$$L^p = L^p(\mathscr{B}, \tau), \qquad 1 \le p \le \infty,$$

be the usual noncommutative Lebesgue spaces and define the noncommutative Hardy space

$$H^p = H^p(\mathscr{B}, \mathscr{E}, \tau)$$

to be the closed subspace of L^p of elements T for which $(1-P)TP = 0$ for all $P \in \mathscr{E}$. In particular $L^\infty = \mathscr{B}$ and $H^\infty = \mathscr{A}$. Let P_0 be the orthogonal projection from L^2 onto H^2 and $\mathscr{F} = \{P_0\}$. By [**13**], with $\mathscr{F} = \{P_0\}$ property (I) holds, that is,

$$\operatorname{dist}(T, H^\infty) = \|(1-P_0)TP_0\|.$$

(Here T acts by left multiplication.)

Assuming $\mathscr{E}$ is well-ordered, (II) is valid by [**13**] again. The theorem shows that $(L^\infty, H^\infty, P_0)$ has the lifting property.

(2) is known and was shown by Arocena and Cotlar [**2**]. However our proof is different from theirs and more operator theoretic. In (2) if U is a bilateral shift with multiplicity 1, then the lifting theorem is classical and was shown by Cotlar and Sadosky (cf. [**4**]). We can show this using (3) and [**3**, Proposition 5.1].

4. Applications. In this section, using the theorem we obtain the partial generalizations of the Helson-Szegö theorem and a theorem of Koosis as in [**4** or **10**].

COROLLARY 1. *Let* $(\mathscr{B},\mathscr{A},\mathscr{F})$ *satisfy the conditions* (I) *and* (II). *Let* W *be a positive operator in* $\mathscr{B}$. *Then there exists a constant* C *such that for every* $P\in\mathscr{F}$.

$$(Wf,f)\le C(W(f+g),f+g)$$

for all $f\in PK$ *and all* $g\in(1-P)K$ *if and only if there exists a nonzero operator* D *in* $\mathscr{A}$ *such that*

$$\|W^{-1/2}(W+D)W^{-1/2}\|\le\rho<1,$$

where $W^{-1/2}$ *is defined to be the* (*possibly unbounded*) *inverse of* $W^{1/2}$ *restricted to the orthocomplent of the kernel of* $W^{1/2}$.

PROOF. $(Wf,f)\le C(W(f+g),f+g)$ if and only if

$$|(Wf,g)|^2\le(1-C^{-1})(Wf,f)(Wg,g).$$

Hence the weighted norm inequality with C holds if and only if $\mathscr{T}=[T_{ij}]$ in $[\mathscr{B}]$ is positive on $PK\oplus(1-P)K$ for every $P\in\mathscr{F}$ where $T_{11}=\rho W$, $T_{22}=T_{12}=T_{21}=W$ and $\rho=1-C^{-1}$. By the theorem, if the weighted norm inequality holds then there exists $\tilde{\mathscr{T}}$ in $\mathscr{T}+[\mathscr{A}]_0$ which is positive on $K\oplus K$. Hence there exists an operator D in $\mathscr{A}$ such that

$$|((W+D)F,G)|^2\le\rho(WF,F)(WG,G)$$

for any F, $G\in K$. This implies $\|W^{-1/2}(W+D)W^{-1/2}\|\le\rho$. The converse is not difficult.

COROLLARY 2. *Suppose that* $(\mathscr{B},\mathscr{A},\mathscr{F})$ *satisfies the conditions* (I) *and* (II). *Let* W *be a positive operator in* $\mathscr{B}$. *Then there exists a nonzero positive operator* $U\le W$ *such that for every* $P\in\mathscr{F}$

$$(Uf,f)\le(W(f+g),f+g)$$

for all $f\in PK$ *and all* $g\in(1-P)K$ *if and only if there exists a nonzero operator* D *in* $\mathscr{A}$ *such that*

$$\|W^{-1/2}(W+D)W^{-1/2}\|\le 1$$

and $W\ne(W^{-1/2}(W+D))^*(W^{-1/2}(W+D))$.

PROOF. The part of 'only if' can be proved as in Corollary 1. If there exists a nonzero positive operator U then $\mathscr{T}=[T_{ij}]\in[\mathscr{B}]$ is positive on $PK\oplus(1-P)K$ for every $P\in\mathscr{F}$ where $T_{11}=W-U$, $T_{22}=T_{12}$, $T_{21}=W$. By the theorem, there exists an operator D in $\mathscr{A}$ such that

$$|((W+D)F,G)|^2\le((W-U)F,F)(WG,G)$$

for any F, $G\in K$. This implies $\|W^{-1/2}(W+D)W^{-1/2}\|\le 1$ and

$$\|W^{-1/2}(W+D)F\|^2=\sup_G\frac{|((W+D)F,G)|^2}{(WG,G)}\le((W-U)F,F)\le(WF,F)$$

(see [**1**, Chapter I]). Thus $W-(W^{-1/2}(W+D))^*(W^{-1/2}(W+D))\le U$. For the converse, $U=W-(W^{-1/2}(W+D))^*(W^{-1/2}(W+D))$ satisfies the weighted norm inequalities.

By (2) in §3, Corollary 1 is the Helson-Szegö theorem [**6**] when the weight function W is bounded. Similarly Corollary 2 shows the Koosis theorem [**7**] when the weight function W is bounded. However in the classical case if there exists a nonzero operator D in $\mathscr{A}$ such that $\|W^{-1/2}(W+D)W^{-1/2}\|\le 1$ then $W\ne(W^{-1/2}(W+D))^*(W^{-1/2}(W+D))$.

REFERENCES

1. T. Ando, *Topics on operator inequalities*, Lecture Note, Sapporo, 1978.
2. R. Arocena and M. Cotlar, *Dilation of generalized Toeplitz kernels and some vectorial moment and weighted problems*, Harmonic Analysis (F. Ricci and G. Weiss, eds.), Lecture Notes in Math., vol. 908, Springer-Verlag, Berlin, Heidelberg and New York, 1982, pp. 169–186.
3. W. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), 208–233.
4. M. Cotlar and C. Sadosky, *On the Helson-Szegö theorem and a related class of modified Toeplitz kernels*, Proc. Sympos. Pure Math. vol. 35, Amer. Math. Soc., Providence, R. I., 1979, pp. 383–407.
5. R. Gellar and L. Page, *Inner-outer factorizations of operators*, J. Math. Anal. Appl. **61** (1977), 151–158.
6. H. Helson and G. Szegö, *A problem in prediction theory*, Ann. Mat. Pura Appl. **51** (1960), 107–138.
7. P. Koosis, *Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques*, C. R. Acad. Sci. Paris **291** (1980), 255–257.
8. P. S. Muhly, *Compact operators in the commutant of a contraction*, J. Funct. Anal. **8** (1971), 197–224.
9. T. Nakazi, *Norms of Hankel operators and uniform algebras*, Trans. Amer. Math. Soc. **299** (1987), 573–580.
10. T. Nakazi and T. Yamamoto, *A lifting theorem and uniform algebras*, Trans. Amer. Math. Soc. **305** (1988), 79–94.
11. Z. Nehari, *On bounded bilinear forms*, Ann. of Math. **65** (1957), 153–162.
12. L. B. Page, *Bounded and compact vectorial Hankel operators*, Trans. Amer. Math. Soc. **150** (1970), 529–539.
13. S. C. Power, *Factorizations in analytic operator algebras*, J. Funct. Anal. **67** (1986), 413–432.
14. B. Sz.-Nagy and C. Foias, *Dilation des commutants d'opérateurs*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), A493–A495.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060, JAPAN