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FUNCTIONS NOT VANISHING ON TRIVIAL GLEASON PARTS OF DOUGLAS ALGEBRAS

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ABSTRACT. Let B denote a closed subalgebra of L^∞ containing the space of bounded analytic functions. Let $M(B)$ denote the maximal ideal space of B. Let f be a function in B such that f does not vanish on any Gleason part consisting of a single point. We show that if g is a function in B such that $|g| \leq |f|$ on $M(B)$, then $g/f \in B$.

1. Introduction. Let B denote a closed subalgebra of L^∞ containing H^∞. The analytic structure of the maximal ideal space of B, $M(B)$, can be best understood through the work of K. Hoffman. From his work [**9**, Theorem 4.3] we know that each nontrivial Gleason part of H^∞ is an analytic disk. If such a part is contained in $M(B)$, then the functions in B are holomorphic with respect to the analytic structure of the disk. Thus, we can talk about the order of the zero of f at a point x in $M(B)$. If f vanishes identically on the Gleason part of x, we say that the order of the zero of f at x is infinite. Recently, C. Guillory, K. Izuchi and D. Sarason [**8**] (see the proof of Corollary 1) showed that if f is a function in $H^\infty + C$ which does not vanish identically on any nontrivial Gleason part of $H^\infty + C$, then f is a product of two functions: one is invertible in $H^\infty + C$ and the other is a finite product of interpolating Blaschke products. From this, they proved a division theorem which was also shown in [**3**]. Here we generalize both these results to arbitrary closed subalgebras of L^∞. The main result in this paper is the following generalization of Corollary 1 in [**8**], offering one possible answer to a question raised in [**8**].

THEOREM. *Let B be a closed subalgebra of L^∞ containing H^∞. Let f be a function in B such that f does not vanish at any trivial point in $M(B)$. If g is a function in B such that $|g| \leq |f|$ on $M(B)$, then f divides g in B.*

By trivial point we mean any point x in $M(B)$ such that $P(x) = \{x\}$.

2. Preliminaries. We let B denote any Douglas algebra, that is, a closed subalgebra of L^∞ containing H^∞. The space $M(B)$ may be identified with a closed subset of $M(H^\infty)$ [**11**, pp. 64–65]. It is known that each Gleason part of H^∞ is either contained in or disjoint from $M(B)$.

The pseudohyperbolic distance between two points x and y in $M(B)$ is defined to be

$$\rho_B(x,y) = \sup\{|f(y)|: f \in B,\ ||f||_\infty \leq 1,\ f(x) = 0\}.$$

If z and w are points in the open unit disk D then $\rho(z,w) = |z-w|/|1-\overline{w}z|$. For $x \in M(B)$, let $P_B(x) = \{y \in M(B): \rho_B(x,y) < 1\}$ denote the Gleason part of B

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containing x. Using the preceding comments and the Chang-Marshall Theorem [**5**, **10**], one can show that if $x \in M(B)$, then $P_B(x) = P_{H^\infty}(x)$. We will write $P(x)$ for $P_{H^\infty}(x)$. If $x \in M(H^\infty)$, Hoffman constructs an analytic map L_x of $\mathbf{D}$ onto $P(x)$. If $P(x)$ is nontrivial, then L_x is one-one.

For a function f in B we let $Z_B(f) = \{x \in M(B): f(x) = 0\}$. In case $f \in H^\infty$, we write $Z(f)$ for $Z_{H^\infty}(f)$. If b is a Blaschke product with zero sequence $\{z_n\}$, then b is an interpolating Blaschke product if

$$\inf_n \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z}_k z_n} \right| > 0.$$

Note that

$$(1 - |z_n|^2)|b'(z_n)| = \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z}_k z_n} \right|.$$

The Chang-Marshall Theorem [**5**, **10**] states that a Douglas algebra B is generated by H^∞ together with the complex conjugates of the interpolating Blaschke products invertible in B. The closed subalgebra of B containing functions in B whose complex conjugates also lie in B is denoted QB. We let $QA_B = QB \cap H^\infty$. The algebra generated by inner functions invertible in B and their complex conjugates is denoted by C_B. For $t \in M(QB)$ the set $E_t = \{x \in M(L^\infty): x(q) = t(q)$ for all $q \in QB\}$ is called a QB level set.

Chang [**6**] showed that $B = H^\infty + C_B$ and $B \cap \overline{B} = H^\infty \cap \overline{B} + C_B$. We use this to prove a result about weak peak sets. Recall that if X is a compact Hausdorff space and $A \subseteq C(X)$ is a function algebra on X, then a closed set $E \subset X$ is a weak peak set for A if for any open neighborhood U of E, there exists $f \in A$ such that $||f|| = 1$, $f|E = 1$ and $|f(x)| < 1$ for $x \in X \backslash U$.

3. Douglas algebras.

LEMMA 1. *Let $\{m_\alpha\}$ be a net of points in $M(H^\infty)$ converging to m and let $z \in \mathbf{D}$. Then $L_{m_\alpha}(z)$ converges to $L_m(z)$.*

PROOF. See [**9**, Theorem 4.3].

The following lemma appears in Budde's Doctoral Disseration [**4**] and in Abrams and Weiss' paper [**1**].

LEMMA 2. *Let m be a point in $M(H^\infty)$. Then $\overline{P(m)}$ contains a trivial point.*

The proof of Theorem 1, below, depends on Lemma 1, Lemma 2 and the work of Hoffman [**9**].

THEOREM 1. *Let B be a Douglas algebra. Let f be an H^∞ function which does not vanish on any trivial Gleason part of $M(B)$. Then $f = uh$, where u is a finite product of interpolating Blaschke products and h is an H^∞ function invertible in B.*

PROOF. We claim that there is an open set U in $M(H^\infty)$ containing $M(B)$ such that whenever $x \in U \cap Z(f)$, then f has a zero of finite order at x. If this were not true, then for each neighborhood $\mathcal{O}_\alpha$ of $M(B)$ we could find a point $x_\alpha \in \mathcal{O}_\alpha \cap Z(f)$ such that f has a zero of infinite order at x_α. Now $f \circ L_{x_\alpha}$ is analytic on $\mathbf{D}$ and has a zero of infinite order at 0. Since L_{x_α} maps $\mathbf{D}$ onto $P(x_\alpha)$, we see that f vanishes

on $P(x_\alpha)$. Since $M(H^\infty)$ is compact, we can choose a subnet of $\{x_\alpha\}$ converging to a point x in $M(B)$. Without loss of generality, we may assume $x_\alpha \to x$. We claim that f vanishes on $P(x)$. First note that by our assumptions on f, the point x is nontrivial. By Lemma 1, $L_{x_\alpha}(z) \to L_x(z)$ for all $z \in \mathbf{D}$. The definition of weak* convergence implies that $f(L_{x_\alpha}(z)) \to f(L_x(z))$ for all $z \in D$. Thus, $f \circ L_x$ vanishes on $\mathbf{D}$. Since L_x maps $\mathbf{D}$ onto P_x, we see that f must vanish on $P(x)$. By the comments preceding the proof of Lemma 1, we know that $\overline{P(x)} \subseteq M(B)$. By Lemma 2, f must vanish at a trivial point in $\overline{P(x)}$. This is impossible. Hence such a set U exists. Let V be an open set in $M(H^\infty)$ satisfying

$$M(B) \subseteq V \subseteq \overline{V} \subseteq U,$$

and let b_1 be the Blaschke product having $Z(f) \cap \mathbf{D} \cap V$ as zero set (with the order of the zero equal to that of the zero of f). Write $f = bk$, where b is a Blaschke product and k is an H^∞ function that has no zeros on $\mathbf{D}$. We shall show that k is invertible in B. To this end, let $x \in M(B)$. Now k has roots of all orders. Thus if $x(k) = 0$, then k has a zero of infinite order at x. Hence k vanishes on $P(x)$. Again using Lemma 2 and the comments preceding the proof of Lemma 1, we see that this would force k to vanish on a trivial point in $M(B)$. Hence k is invertible in $M(B)$.

Let $b_2 = b/b_1$. If $x \in M(B)$ is such that $x(b_2) = 0$, then $x(f) = 0$. Now $x \in M(B) \subseteq V$, so f (hence b_2) has a zero of finite order at x. Thus [**9**, Theorem 5.3] x lies in the closure of an interpolating subsequence of the zero sequence of b_2. But $Z(b_2) \cap \mathbf{D} \subseteq M(H^\infty) - V$ and $x \in V$, which is impossible. Thus $b_2 k$ is invertible in B.

Finally, consider b_1. Let $\{z_k\}$ denote the zero sequence of b_1. Let N_k denote the number of zeros of b_1 a pseudohyperbolic distance less than $\frac{1}{2}$ from z_k. If N_k were unbounded, there would exist a subsequence of $\{z_k\}$ along which N_k tends to ∞. Using the lower semicontinuity of ρ on $M(H^\infty) \times M(H^\infty)$ [**9**, Theorem 6.2], one can show that b_1 would have to vanish identically on any part containing a cluster point of this subsequence. But any cluster point of $\{z_k\}$ would lie in $\overline{V}$, hence in U. Thus N_k is bounded. Decompose the zero sequence of b_1 into finitely many sequences such that the terms of each sequence are at a pseudohyperbolic distance of at least $\frac{1}{2}$ from the other terms of the (same) sequence. Form the Blaschke products corresponding to these sequences. Then b_1 is a product of these (finitely many) Blaschke products. We claim that each of these factors of b_1 is an interpolating Blaschke product. Let c denote one of these factors. We shall first show that if $x \in U$ and $x(c) = 0$, then c has a zero of order 1 at x. If this were not true, we could factor $c = c_1 c_2$ with both c_1 and c_2 vanishing at x. Both c_1 and c_2 have zeros of finite order at x. By [**9**, Theorem 5.3], x lies in the closure of interpolating subsequences of $Z(c_1) \cap \mathbf{D}$ and $Z(c_2) \cap \mathbf{D}$. By [**9**, Theorem 6.1] the pseudohyperbolic distance between $Z(c_1) \cap \mathbf{D}$ and $Z(c_2) \cap \mathbf{D}$ must be zero. By our choice of the factors of b, this is impossible. Now suppose that c is not interpolating. Let $\{z_{n_k}\}$ be a subsequence of the zero sequence of c such that $(1 - |z_{n_k}|^2)|c'(z_{n_k})| \to 0$. Let $x \in \overline{\{z_{n_k}\}}$. Then $x \in U$. Let $\{z_{n_\alpha}\}$ be a subnet of $\{z_{n_k}\}$ such that $z_{n_\alpha} \to x$. By Lemma 1, $\hat{c} \circ L_{z_{n_\alpha}} \to \hat{c} \circ L_x$ pointwise boundedly on $\mathbf{D}$. Hence $(\hat{c} \circ L_{z_{n_\alpha}})'(0) \to (\hat{c} \circ L_x)'(0)$. But $(\hat{c} \circ L_{z_{n_\alpha}})'(0) = (1 - |z_{n_\alpha}|^2)|c'(z_{n_\alpha})|$. Thus $(\hat{c} \circ L_x)'(0) = 0$, which contradicts the fact that c has a zero of order 1 at x.

Thus there exists $\delta > 0$ such that $\inf_{z_n \in Z(c) \cap \mathbf{D}} (1 - |z_n|^2)|c'(z_n)| \geq \delta$. Therefore c is interpolating. Let $u = b_1$ and $h = b_2 k$. Then $f = uh$, where u is a finite product of interpolating Blaschke products, and h is an H^∞ function invertible in B.

To prove the main result of this paper, we need to use some facts about level sets of Douglas algebras.

LEMMA 3. *Let B be a Douglas algebra. Let E be a QB level set. Then $H^\infty|E = B|E$.*

PROOF. By [**6**], $B = H^\infty + QB$. Thus

$$H^\infty|E = (H^\infty + QB)|E = B|E.$$

It is well known [**7**, p. 60] that if E is a QB level set, then E is a weak peak set for B. Hence $B|E$ is closed (see [**7**, pp. 56–60]). By Lemma 3, $H^\infty|E$ is closed. Note that this forces E to be a weak peak set for H^∞ [**7**, p. 65, Exercise 14]. Let $H_E^\infty = \{f \in L^\infty : f|E \in H^\infty|E\}$. Since $H^\infty|E$ is closed, H_E^∞ is a Douglas algebra. Furthermore [**7**, p. 39, **2**, p. 17] $M(H_E^\infty) = M(L^\infty) \cup M(H^\infty|E)$. To prove the main result, we shall first prove it in the case where $B = H_E^\infty$ and E is a weak peak set for H^∞.

LEMMA 4. *Let E be a weak peak set for H^∞. Let f be a function in H_E^∞ which does not vanish on any trivial part of $M(H_E^\infty)$. Let g be a function in H_E^∞ such that $|g| \leq |f|$ on $M(H_E^\infty)$. Then f divides g in H_E^∞.*

PROOF. By the definition of H_E^∞, there exists $h \in H^\infty$ such that $h|E = f|E$. To apply Theorem 1 to h, we need to know that h does not vanish on any trivial part of $M(H_E^\infty)$. If $x \in M(H^\infty|E)$, then [**7**, p. 39] there is a representing measure for x supported on E. Hence $x(f) = x(h)$. As in the proof of Theorem 1, it follows from Lemma 2 that the outer factor d of h has the property that $d|E$ is invertible in $H^\infty|E$. Let I be the inner factor of h. Then I does not vanish on any trivial part of $M(H^\infty|E) \cup M(L^\infty) = M(H_E^\infty)$. Applying Theorem 1, we may write $I = bk$, where b is a finite product of interpolating Blaschke products and k is invertible in H_E^∞. Since I is invertible in L^∞, replacing g by εg for a small positive constant ε, we may assume that $|g| \leq |I|$ on $M(H_E^\infty)$. By [**3**, Lemma 3 or **8**], I divides g in H_E^∞. Since f is invertible in L^∞, $g/f \in L^\infty$. Since $f|E = h|E$, we have $g/f|E = g/h|E = (g/I)(1/d)|_E \in H^\infty|_E$. Thus f divides g in H_E^∞.

The proof of the main result of this paper will follow easily from Lemma 4 and Shilov's Theorem [**7**, p. 60].

THEOREM 2. *Let B be a Douglas algebra. Let f be any function in B which does not vanish on any trivial Gleason part of $M(B)$. Let g be a function in B such that $|g| \leq |f|$ on $M(B)$. Then f divides g in B.*

PROOF. Let E be a QB level set. By Lemma 3, $f|E \in H^\infty|E$ and $g|E \in H^\infty|E$. By [**7**, p. 39] if $x \in M(H^\infty|E)$, then $x \in M(B)$. Hence $|g| \leq |f|$ on $M(H_E^\infty)$. By Lemma 4, $g/f|E \in H^\infty|E = B|E$. Since this holds for every QB level set E, by Shilov's theorem [**7**, p. 60] $g/f \in B$ as desired.

It is easy to see from what has been done here that, if g is a function in B such that every zero of f is a zero of g of at least as high an order, then f divides g in

B (use the proofs above and Theorem 1 of [**8**]). However, I have been unable to prove the following:

> Let f be a unimodular function in B such that f does not vanish identically on any nontrivial part in $M(B)$. If g is a function in B such that every zero of f is a zero of g of at least as high an order, then g is divisible by f in B.

Let f be a function in $H^\infty + C$. Then f vanishes on a nontrivial part in $M(H^\infty + C)$ if and only if f vanishes on a trivial part. It is not known whether this is true in an arbitrary Douglas algebra B.

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