FUNCTIONS NOT VANISHING ON TRIVIAL GLEASON PARTS OF DOUGLAS ALGEBRAS

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Abstract. Let $B$ denote a closed subalgebra of $L^\infty$ containing the space of bounded analytic functions. Let $M(B)$ denote the maximal ideal space of $B$. Let $f$ be a function in $B$ such that $f$ does not vanish on any Gleason part consisting of a single point. We show that if $g$ is a function in $B$ such that $|g| \leq |f|$ on $M(B)$, then $g/f \in B$.

1. Introduction. Let $B$ denote a closed subalgebra of $L^\infty$ containing $H^\infty$. The analytic structure of the maximal ideal space of $B$, $M(B)$, can be best understood through the work of K. Hoffman. From his work [9, Theorem 4.3] we know that each nontrivial Gleason part of $H^\infty$ is an analytic disk. If such a part is contained in $M(B)$, then the functions in $B$ are holomorphic with respect to the analytic structure of the disk. Thus, we can talk about the order of the zero of $f$ at a point $x$ in $M(B)$. If $f$ vanishes identically on the Gleason part of $x$, we say that the order of the zero of $f$ at $x$ is infinite. Recently, C. Guillory, K. Izuchi and D. Sarason [8] (see the proof of Corollary 1) showed that if $f$ is a function in $H^\infty + C$ which does not vanish identically on any nontrivial Gleason part of $H^\infty + C$, then $f$ is a product of two functions: one is invertible in $H^\infty + C$ and the other is a finite product of interpolating Blaschke products. From this, they proved a division theorem which was also shown in [3]. Here we generalize both these results to arbitrary closed subalgebras of $L^\infty$. The main result in this paper is the following generalization of Corollary 1 in [8], offering one possible answer to a question raised in [8].

Theorem. Let $B$ be a closed subalgebra of $L^\infty$ containing $H^\infty$. Let $f$ be a function in $B$ such that $f$ does not vanish at any trivial point in $M(B)$. If $g$ is a function in $B$ such that $|g| \leq |f|$ on $M(B)$, then $f$ divides $g$ in $B$.

By trivial point we mean any point $x$ in $M(B)$ such that $P(x) = \{x\}$.

2. Preliminaries. We let $B$ denote any Douglas algebra, that is, a closed subalgebra of $L^\infty$ containing $H^\infty$. The space $M(B)$ may be identified with a closed subset of $M(H^\infty)$ [11, pp. 64-65]. It is known that each Gleason part of $H^\infty$ is either contained in or disjoint from $M(B)$.

The pseudohyperbolic distance between two points $x$ and $y$ in $M(B)$ is defined to be

$$\rho_B(x, y) = \sup\{|f(y)|: f \in B, ||f||_\infty \leq 1, f(x) = 0\}.$$ 

If $z$ and $w$ are points in the open unit disk $D$ then $\rho(z, w) = |z - w|/|1 - \bar{w}z|$. For $x \in M(B)$, let $P_B(x) = \{y \in M(B): \rho_B(x, y) < 1\}$ denote the Gleason part of $B$.

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containing $x$. Using the preceding comments and the Chang-Marshall Theorem [5, 10], one can show that if $x \in M(B)$, then $P_B(x) = P_{H^\infty}(x)$. We will write $P(x)$ for $P_{H^\infty}(x)$. If $x \in M(H^\infty)$, Hoffman constructs an analytic map $L_x$ of $D$ onto $P(x)$. If $P(x)$ is nontrivial, then $L_x$ is one-one.

For a function $f$ in $B$ we let $Z_B(f) = \{x \in M(B): f(x) = 0\}$. In case $f \in H^\infty$, we write $Z(f)$ for $Z_{H^\infty}(f)$. If $b$ is a Blaschke product with zero sequence $\{z_n\}$, then $b$ is an interpolating Blaschke product if

$$\inf_n \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z_k}z_n} \right| > 0.$$

Note that

$$(1 - |z_n|^2)|b'(z_n)| = \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z_k}z_n} \right|.$$

The Chang-Marshall Theorem [5, 10] states that a Douglas algebra $B$ is generated by $H^\infty$ together with the complex conjugates of the interpolating Blaschke products invertible in $B$. The closed subalgebra of $B$ containing functions in $B$ whose complex conjugates also lie in $B$ is denoted $QB$. We let $QA_B = QB \cap H^\infty$. The algebra generated by inner functions invertible in $B$ and their complex conjugates is denoted by $C_B$. For $t \in M(QB)$ the set $E_t = \{x \in M(L^\infty): x(q) = t(q) \text{ for all } q \in QB\}$ is called a $QB$ level set.

Chang [6] showed that $B = H^\infty + C_B$ and $B \cap \overline{B} = H^\infty \cap \overline{B} + C_B$. We use this to prove a result about weak peak sets. Recall that if $X$ is a compact Hausdorff space and $A \subseteq C(X)$ is a function algebra on $X$, then a closed set $E \subseteq X$ is a weak peak set for $A$ if for any open neighborhood $U$ of $E$, there exists $f \in A$ such that $||f|| = 1$, $f|E = 1$ and $|f(x)| < 1$ for $x \in X \setminus U$.

3. Douglas algebras.

**Lemma 1.** Let $\{m_\alpha\}$ be a net of points in $M(H^\infty)$ converging to $m$ and let $z \in D$. Then $L_{m_\alpha}(z)$ converges to $L_m(z)$.

**Proof.** See [9, Theorem 4.3].

The following lemma appears in Budde’s Doctoral Dissertation [4] and in Abrams and Weiss’ paper [1].

**Lemma 2.** Let $m$ be a point in $M(H^\infty)$. Then $\overline{P(m)}$ contains a trivial point.

The proof of Theorem 1, below, depends on Lemma 1, Lemma 2 and the work of Hoffman [9].

**Theorem 1.** Let $B$ be a Douglas algebra. Let $f$ be an $H^\infty$ function which does not vanish on any trivial Gleason part of $M(B)$. Then $f = uh$, where $u$ is a finite product of interpolating Blaschke products and $h$ is an $H^\infty$ function invertible in $B$.

**Proof.** We claim that there is an open set $U$ in $M(H^\infty)$ containing $M(B)$ such that whenever $x \in U \cap Z(f)$, then $f$ has a zero of finite order at $x$. If this were not true, then for each neighborhood $\mathcal{O}_\alpha$ of $M(B)$ we could find a point $x_\alpha \in \mathcal{O}_\alpha \cap Z(f)$ such that $f$ has a zero of infinite order at $x_\alpha$. Now $f \circ L_{x_\alpha}$ is analytic on $D$ and has a zero of infinite order at 0. Since $L_{x_\alpha}$ maps $D$ onto $P(x_\alpha)$, we see that $f$ vanishes
on $P(x_\alpha)$. Since $M(H^\infty)$ is compact, we can choose a subnet of $\{x_\alpha\}$ converging to a point $x$ in $M(B)$. Without loss of generality, we may assume $x_\alpha \to x$. We claim that $f$ vanishes on $P(x)$. First note that by our assumptions on $f$, the point $x$ is nontrivial. By Lemma 1, $L_{x_\alpha}(z) \to L_x(z)$ for all $z \in D$. The definition of weak* convergence implies that $f(L_{x_\alpha}(z)) \to f(L_x(z))$ for all $z \in D$. Thus, $f \circ L_x$ vanishes on $D$. Since $L_x$ maps $D$ onto $P_x$, we see that $f$ must vanish on $P(x)$. By the comments preceding the proof of Lemma 1, we know that $P(x) \subseteq M(B)$. By Lemma 2, $f$ must vanish at a trivial point in $P(x)$. This is impossible. Hence such a set $U$ exists. Let $V$ be an open set in $M(H^\infty)$ satisfying

$$M(B) \subseteq V \subseteq \overline{V} \subseteq U,$$

and let $b_1$ be the Blaschke product having $Z(f) \cap D \cap V$ as zero set (with the order of the zero equal to that of the zero of $f$). Write $f = bk$, where $b$ is a Blaschke product and $k$ is an $H^\infty$ function that has no zeros on $D$. We shall show that $k$ is invertible in $B$. To this end, let $x \in M(B)$. Now $k$ has roots of all orders. Thus if $x(k) = 0$, then $k$ has a zero of infinite order at $x$. Hence $k$ vanishes on $P(x)$. Again using Lemma 2 and the comments preceding the proof of Lemma 1, we see that this would force $k$ to vanish on a trivial point in $M(B)$. Hence $k$ is invertible in $M(B)$.

Let $b_2 = b/b_1$. If $x \in M(B)$ is such that $x(b_2) = 0$, then $x(f) = 0$. Now $x \in M(B) \subseteq V$, so $f$ (hence $b_2$) has a zero of finite order at $x$. Thus $[9, \text{Theorem 5.3}]$ $x$ lies in the closure of an interpolating subsequence of the zero sequence of $b_2$. But $Z(b_2) \cap D \subseteq M(H^\infty) - V$ and $x \in V$, which is impossible. Thus $b_2k$ is invertible in $B$.

Finally, consider $b_1$. Let $\{z_k\}$ denote the zero sequence of $b_1$. Let $N_k$ denote the number of zeros of $b_1$ a pseudohyperbolic distance less than $\frac{1}{2}$ from $z_k$. If $N_k$ were unbounded, there would exist a subsequence of $\{z_k\}$ along which $N_k$ tends to $\infty$. Using the lower semicontinuity of $\rho$ on $M(H^\infty) \times M(H^\infty)$ $[9, \text{Theorem 6.2}]$, one can show that $b_1$ would have to vanish identically on any part containing a cluster point of this subsequence. But any cluster point of $\{z_k\}$ would lie in $\overline{V}$, hence in $U$. Thus $N_k$ is bounded. Decompose the zero sequence of $b_1$ into finitely many sequences such that the terms of each sequence are at a pseudohyperbolic distance of at least $\frac{1}{2}$ from the other terms of the (same) sequence. Form the Blaschke products corresponding to these sequences. Then $b_1$ is a product of these (finitely many) Blaschke products. We claim that each of these factors of $b_1$ is an interpolating Blaschke product. Let $c$ denote one of these factors. We shall first show that if $x \in U$ and $x(c) = 0$, then $c$ has a zero of order 1 at $x$. If this were not true, we could factor $c = c_1c_2$ with both $c_1$ and $c_2$ vanishing at $x$. Both $c_1$ and $c_2$ have zeros of finite order at $x$. By $[9, \text{Theorem 5.3}]$, $x$ lies in the closure of interpolating subsequences of $Z(c_1) \cap D$ and $Z(c_2) \cap D$. By $[9, \text{Theorem 6.1}]$ the pseudohyperbolic distance between $Z(c_1) \cap D$ and $Z(c_2) \cap D$ must be zero. By our choice of the factors of $b$, this is impossible. Now suppose that $c$ is not interpolating. Let $\{z_{n_k}\}$ be a subsequence of the zero sequence of $c$ such that $(1 - |z_{n_k}|^2)c'(z_{n_k}) \to 0$. Let $x \in \{z_{n_k}\}$. Then $x \in U$. Let $\{z_{n_\alpha}\}$ be a subnet of $\{z_{n_k}\}$ such that $z_{n_\alpha} \to x$. By Lemma 1, $\hat{\epsilon} \circ L_{z_{n_\alpha}} \to \hat{\epsilon} \circ L_x$ pointwise boundedly on $D$. Hence $(\hat{\epsilon} \circ L_{z_{n_\alpha}})'(0) \to (\hat{\epsilon} \circ L_x)'(0)$. But $(\hat{\epsilon} \circ L_{z_{n_\alpha}})'(0) = (1 - |z_{n_\alpha}|^2)c'(z_{n_\alpha})$. Thus $(\hat{\epsilon} \circ L_x)'(0) = 0$, which contradicts the fact that $c$ has a zero of order 1 at $x$.  

Thus there exists $\delta > 0$ such that $\inf_{z_n \in Z(c) \cap D} (1 - |z_n|^2)|c'(z_n)| \geq \delta$. Therefore $c$ is interpolating. Let $u = b_1$ and $h = b_2 k$. Then $f = uh$, where $u$ is a finite product of interpolating Blaschke products, and $h$ is an $H^\infty$ function invertible in $B$.

To prove the main result of this paper, we need to use some facts about level sets of Douglas algebras.

**Lemma 3.** Let $B$ be a Douglas algebra. Let $E$ be a QB level set. Then $H^\infty|E = B|E$.

**Proof.** By [6], $B = H^\infty + QB$. Thus

$$H^\infty|E = (H^\infty + QB)|E = B|E.$$  

It is well known [7, p. 60] that if $E$ is a QB level set, then $E$ is a weak peak set for $B$. Hence $B|E$ is closed (see [7, pp. 56-60]). By Lemma 3, $H^\infty|E$ is closed. Note that this forces $E$ to be a weak peak set for $H^\infty$ [7, p. 65, Exercise 14]. Let $H^\infty_E = \{f \in L^\infty : f|E \in H^\infty|E\}$. Since $H^\infty|E$ is closed, $H^\infty_E$ is a Douglas algebra. Furthermore [7, p. 39, 2, p. 17] $M(H^\infty_E) = M(L^\infty) \cup M(H^\infty|E)$. To prove the main result, we shall first prove it in the case where $B = H^\infty_E$ and $E$ is a weak peak set for $H^\infty$.

**Lemma 4.** Let $E$ be a weak peak set for $H^\infty$. Let $f$ be a function in $H^\infty_E$ which does not vanish on any trivial part of $M(H^\infty_E)$. Let $g$ be a function in $H^\infty_E$ such that $|g| \leq |f|$ on $M(H^\infty_E)$. Then $f$ divides $g$ in $H^\infty_E$.

**Proof.** By the definition of $H^\infty_E$, there exists $h \in H^\infty$ such that $h|E = f|E$. To apply Theorem 1 to $h$, we need to know that $h$ does not vanish on any trivial part of $M(H^\infty_E)$. If $x \in M(H^\infty|E)$, then [7, p. 39] there is a representing measure for $x$ supported on $E$. Hence $x(f) = x(h)$. As in the proof of Theorem 1, it follows from Lemma 2 that the outer factor $d$ of $h$ has the property that $d|E$ is invertible in $H^\infty|E$. Let $I$ be the inner factor of $h$. Then $I$ does not vanish on any trivial part of $M(H^\infty|E) \cup M(L^\infty) = M(H^\infty_E)$. Applying Theorem 1, we may write $I = b k$, where $b$ is a finite product of interpolating Blaschke products and $k$ is invertible in $H^\infty_E$. Since $I$ is invertible in $L^\infty$, replacing $g$ by $\varepsilon g$ for a small positive constant $\varepsilon$, we may assume that $|g| \leq |I|$ on $M(H^\infty_E)$. By [3, Lemma 3 or 8], $I$ divides $g$ in $H^\infty_E$. Since $g$ is invertible in $L^\infty$, $g/f \in L^\infty$. Since $f|E = h|E$, we have $g/f|E = g/h|E = (g/I)(1/d)|E \in H^\infty|E$. Thus $f$ divides $g$ in $H^\infty_E$.

The proof of the main result of this paper will follow easily from Lemma 4 and Shilov’s Theorem [7, p. 60].

**Theorem 2.** Let $B$ be a Douglas algebra. Let $f$ be any function in $B$ which does not vanish on any trivial Gleason part of $M(B)$. Let $g$ be a function in $B$ such that $|g| \leq |f|$ on $M(B)$. Then $f$ divides $g$ in $B$.

**Proof.** Let $E$ be a QB level set. By Lemma 3, $f|E \in H^\infty|E$ and $g|E \in H^\infty|E$. By [7, p. 39] if $x \in M(H^\infty|E)$, then $x \in M(B)$. Hence $|g| \leq |f|$ on $M(H^\infty_E)$. By Lemma 4, $g/f|E \in H^\infty|E = B|E$. Since this holds for every QB level set $E$, by Shilov’s theorem [7, p. 60] $g/f \in B$ as desired.

It is easy to see from what has been done here that, if $g$ is a function in $B$ such that every zero of $f$ is a zero of $g$ of at least as high an order, then $f$ divides $g$ in
Let $f$ be a unimodular function in $B$ such that $f$ does not vanish identically on any nontrivial part in $M(B)$. If $g$ is a function in $B$ such that every zero of $f$ is a zero of $g$ of at least as high an order, then $g$ is divisible by $f$ in $B$.

Let $f$ be a function in $H^\infty + C$. Then $f$ vanishes on a nontrivial part in $M(H^\infty + C)$ if and only if $f$ vanishes on a trivial part. It is not known whether this is true in an arbitrary Douglas algebra $B$.

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**BIBLIOGRAPHY**


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