THE FIXED-POINT INDEX AND THE FIXED-POINT THEOREMS OF 1-SET-CONTRACTION MAPPINGS

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ABSTRACT. W. V. Petryshyn [1] studied the fixed-point theorem of 1-set-contraction mappings. This paper gives the definition of the fixed-point index of 1-set-contraction mappings and the concept of semiclosed 1-set-contraction mappings; then we obtain some fixed-point theorems about it.

In this paper we define the fixed-point index of 1-set-contraction mappings and give the concept of semiclosed 1-set-contraction mappings. Hence the fixed-point index of strict-set-contraction mappings (Nussbaum [2]) is extended to 1-set-contraction mappings. We obtain furthermore some fixed-point theorems of semiclosed 1-set-contraction mappings. Let $E$ be a real Banach space, $X$ a nonempty closed convex subset of $E$, $D$ a relative bounded open set with respect to $X$, and $\overline{D}$ and $\partial D$ the closure and boundary of $D$ in $X$ respectively.

Suppose that $T: \overline{D} \to X$ is 1-set-contraction mapping and $\theta \notin (I - T)\partial D$, so there exists $\delta > 0$ such that

$$(1) \quad \inf_{x \in \partial D} \|x - Tx\| \geq \delta.$$

We set $T_k = kT$, where $k \in (1 - \delta/M, 1)$, $M = \sup_{x \in D} \|Tx\| + \delta$. Obviously $T_k$ is a strict-set-contraction mapping. By Amann [3], $T_k$ has fixed-point index $i(T_k, D)$.

DEFINITION 1. The fixed-point index $i(T, D)$ of $T$ on $D$ is defined by $i(T, D) = i(T_k, D)$.

We can prove that $i(T, D)$ is defined uniquely and independent of $T_k$. In fact, suppose that $W_i: \overline{D} \to X$ is a $K_i$-set-contraction mapping $(0 < K_i < 1)$ with

$$(2) \quad \|W_ix - Tx\| < \delta \quad x \in \partial D, \ i = 1, 2.$$

We make a homotopic mapping on $\overline{D}$ as follows

$$H(t, x) = tW_1x + (1 - t)W_2x \quad x \in \overline{D}, \ t \in [0, 1].$$

$H_i: \overline{D} \to X$ is a $\lambda$-set-contraction mapping, where $\lambda = \max\{k_1, k_2\}$. For every $x \in \partial D$

$$\|x - H(t, x)\| = \|x - tW_1x - (1 - t)W_2x\| \geq \|x - Tx\| - t\|W_1x - W_2x\| - (1 - t)\|W_2x - W_1x\| > \delta - t\delta - (1 - t)\delta = 0.$$
By Proposition 2.1(3) of [3], \( i(H(t, x), D) = \text{const.} \). Then
\[
i(H(1, x), D) = i(H(0, x), D),
\]
i.e., \( i(W_1, D) = i(W_2, D) \). This equality shows that \( i(T, D) \) is independent of \( T_k \).

\( i(T, D) \) satisfies three axioms of the fixed-point index:

**Theorem 1.** (I) Let \( Tx = x_0 \forall x \in \overline{D} \);
\[
i(T, D) = \begin{cases} 1, & x_0 \in D, \\ 0, & x_0 \notin D. \end{cases}
\]

(II) Suppose that \( D, D_1, \) and \( D_2 \) are open subsets of \( X, D_1 \cup D_2 \subset D \) and \( D_1 \cap D_2 = \emptyset, T \) has no fixed-point on \( \overline{D} \setminus (D_1 \cup D_2) \). Then \( i(T, D) = i(T, D_1) + i(T, D_2) \).

(III) Suppose that the mapping \( H: [0,1] \times D \to X \) is continuous and \( \theta \notin (I - H(t, x))(0,1] \times \partial D \). Suppose that for every \( t \in [0,1], H(t, x): \overline{D} \to X \) is a 1-set-contraction mapping. Finally suppose that \( H(t, x): [0,1] \times D \) is uniformly continuous with respect to \( x \in \overline{D} \). Then \( i(H(1, x), D) = i(H(0, x), D) \) and \( i(H(t, x), D) \) is independent of \( t \).

**Proof.** (I) Since \( T \) is a constant mapping, \( T \) must be a 0-set-contraction mapping. Hence
\[
i(T, D) = \begin{cases} 1, & x_0 \in D, \\ 0, & x_0 \notin D. \end{cases}
\]

(II) As discussed in Definition 1, we set
\[
\delta = \inf \{ \| x - Tx \| \mid x \in \overline{D} \setminus (D_1 \cup D_2) \}.
\]

Suppose that \( W: \overline{D} \to X \) is a strict-set-contraction mapping such that
\[
(3) \quad \| Tx - Wx \| < \delta, \quad x \in \overline{D} \setminus (D_1 \cup D_2).
\]

By Definition 1,
\[
i(T, D) = i(W, D) \quad \text{and} \quad (T, D_j) = i(W, D_j) \quad (j = 1, 2).
\]

\( W \) has no fixed point on \( \overline{D} \setminus (D_1 \cup D_2) \).

In fact, if there is \( x_0 \in \overline{D} \setminus (D_1 \cup D_2) \) such that \( x_0 = Wx_0 \), then
\[
\delta \leq \| x_0 - Tx_0 \| = \| x_0 - Wx_0 + Wx_0 - Tx_0 \| = \| Wx_0 - Tx_0 \|,
\]

which contradicts (3).

By Proposition 2.1 of [3], \( i(W, D) = i(W, D_1) + i(W, D_2) = i(T, D_1) + i(T, D_2) \); then \( i(T, D) = i(T, D_1) + i(T, D_2) \).

(III) By the definition of \( H(t, x) \), there are \( \delta > 0 \) and \( M > 0 \) such that
\[
\| x - H(t, x) \| \geq \delta > 0 \quad t \in [0,1], x \in \partial D,
\]

and \( \| H(t, x) \| < M, t \in [0,1], x \in \overline{D} \).

Set \( k \in (1 - \delta/2M, 1) \),
\[
G(t, x) = kH(t, x): [0,1] \times \overline{D} \to X,
\]

\[
\| H(t, x) - G(t, x) \| \leq (1 - k)M < \delta/2, \quad t \in [0,1], x \in \overline{D};
\]

then
\[
\| G(t, x) - x \| \geq \| x - H(t, x) \| - \| H(t, x) - G(t, x) \| > \delta - \delta/2 > 0.
\]
For every $t \in [0,1]$, obviously $G(t, x) : \overline{D} \to X$ is a $k$-set-contraction and $G(\cdot, x) : [0,1] \to X$ is uniformly continuous with respect to $x \in \overline{D}$.

By Proposition 2.1(3) of [3],
\[ i(G(t,x), D) = \text{const.} \quad (t \in [0,1]). \]

According to Definition 1
\[ i(H(t,x), D) = i(G(t,x), D), \]
then
\[ i(H(1,x), x) = i(H(0,x), D). \]

**DEFINITION 2.** $T$ is called a semiclosed 1-set-contraction mapping, if $T$ is 1-set-contraction and $I - T$ is closed.

Let us state a definition of semicompact mapping [6].

$T : E \to E$ is said to be semicompact if for each bounded sequence $\{x_n\}$ in $E$ such that $x_n - T(x_n) \to y$ for some $y$ in $E$, there exists a convergent subsequence.

Obviously, semicompact 1-set-contraction mapping $\Rightarrow$ semiclosed 1-set-contraction mapping.

**THEOREM 2 (SOLVABILITY).** Let $E$ be a real Banach space, $X$ be a closed convex subset of $E$, $D$ be a bounded open set with respect to $X$.

Suppose that $T : \overline{D} \to X$ is semiclosed 1-set-contraction mapping, $\theta \notin (I - T)\partial D$, $i(T, D) \neq 0$. Then $T$ has a fixed point in $D$.

**PROOF.** Suppose that $k_n \in (0,1)$ and $k_n \to 1$. We set $W_n = k_nT$, $W_n : \overline{D} \to X$ is $k_n$-set-contraction. Since $\sup_{x \in \overline{D}} \|Tx\| < +\infty$,
\[ \|Tx - W_nx\| = \|Tx - k_nTx\| = (1 - k_n)\|Tx\| \to 0 \quad (n \to \infty), \]
then there is $N$, such that for every $n > N$
\[ \|Tx - W_nx\| < \delta \quad \text{where} \quad 0 < \delta < \inf_{x \in \partial D} \|x - Tx\|. \]

By the definition and our assumptions, $i(T, D) = i(W_n, D) \neq 0$. By Proposition 2.1(1) of [3], $W_n$ has a fixed point $x_n \in D$, i.e., $x_n = W_nx_n$. Hence for every $n > N$
\[ \|x_n - Tx_n\| = \|x_n - W_nx_n - Tx_n + W_nx_n\| = \|W_nx_n - Tx_n\| \to 0 \quad (n \to \infty). \]

Then $x_n - Tx_n \to \theta \ (n \to \infty)$.

Since $I - T$ is closed, so $\theta \in (I - T)\overline{D}$, there is $x_0 \in \overline{D}$ such that $x_0 = Tx_0$. Because $\theta \notin (I - T)\partial D$, there exists $x_0 \in D$ such that $x_0 = Tx_0$. Q.E.D.

**THEOREM 3.** Let $E$ be a real Banach space, $X$ be a closed convex set in $E$, $D$ be a bounded open set in $X$. Suppose that $T : \overline{D} \to X$ is a semiclosed 1-set-contraction mapping, and that $x_0 \in D$ is such that
\[(a) \quad x \neq tTx + (1 - t)x_0, \quad x \in \partial D, \quad t \in (0,1). \]

Then $T$ has a fixed point in $\overline{D}$.

**PROOF.** If $T$ has a fixed point on $\partial D$, the conclusion of this theorem obviously holds.
If $T$ has no fixed point on $\partial D$, then $i(T, D)$ has meaning and
\[ x \neq tTx + (1 - t)x_0 \quad x \in \partial D, \ t \in [0, 1]. \]
Set
\[ H(t, x) = tTx + (1 - t)x_0. \]
We know easily that $H(t, x)$ satisfies the conditions of Theorem 1(III), so that
\[ i(T, D) = i(x_0, D) = 1. \]
By Theorem 2, $T$ has a fixed point in $D$. Q.E.D.

If $T(\partial D) \subset D, x_0 \in D$, for arbitrary $x \in \partial D, 0 < t \leq 1$, we have that
\[ tTx + (1 - t)x_0 \neq x, \quad x \in \partial D. \]
Then the condition (a) is satisfied. We have the following consequences:

**COROLLARY 1.** If $T(\partial D) \subset D$, then $T$ has a fixed point in $D$, where $D$ is convex.

**COROLLARY 2.** If $x_0 = \theta \in D$ and $x \neq Tx, x \in \partial D, t \in (0, 1]$, then $i(T, D) = 1$, and $T$ has a fixed point in $D$.

**THEOREM 4.** Let $E$ be a real Banach space, $X$ be a wedge in $E$ (Amann [5]), and $D$ be a bounded open set in $X$. Suppose that $T: \overline{D} \to X$ is a semiclosed 1-set-contraction mapping and $x_0 \in X, x_0 \neq \theta$ such that
\[ x \neq Tx + \lambda x_0, \quad \lambda \geq 0, \ x \in \partial D, \]
then $i(T, D) = 0$.

**PROOF.** Since $\overline{D}$ is bounded, there exists $\lambda_0 > 0$ such that $\forall \lambda \geq \lambda_0, \lambda x_0 \notin D$. We set
\[ H_{\lambda}(t, x) = (1 - t)Tx + t\lambda x_0, \quad \lambda \geq \lambda_0, \ t \in [0, 1], \ x \in \overline{D}. \]
For arbitrary $t \in [0, 1]$ and $\lambda \geq \lambda_0, H_{\lambda}(t, \cdot) : \overline{D} \to X$ is a 1-set-contraction mapping.

We can prove that $H_{\lambda}(t, x) \neq x$, where $x \in \partial D, t \in [0, 1], \lambda \geq \lambda_1$ for some $\lambda_1 \geq \lambda_0$, in fact, if the present inequality does not hold, then there are $\lambda_0 \to \infty$ and corresponding $x_n \in \partial D$ and $t_n \in [0, 1]$ such that
\[ x_n = (1 - t_n)Tx_n + t_n\lambda nx_0. \]
Since the set $\{x_n - (1 - t_n)Tx_n\}$ is bounded (and hence $\{t_n\lambda_n\}$ is bounded), we may suppose that $\lambda_n t_n \to l_0 < +\infty$; hence $t_n \to 0 \ (n \to \infty)$, and
\[ x_n - Tx_n = [x_n - (1 - t_n)Tx_n] - t_nTx_n \to l_0x_0. \]
Because $I - T$ is closed, $l_0x_0 \in (I - T)\partial D$. Then there is $x^* \in \partial D$ such that
\[ (I - T)x^* = l_0x_0, \]
which contradicts our assumptions. By Theorem 1 and $\lambda x_0 \notin \overline{D}$ ($\lambda \geq \lambda_0$), we obtain
\[ i(T, D) = i(\lambda x_0, D) = 0 \quad Q.E.D. \]
THEOREM 5. Let $E$ be a real Banach space, $X$ be a wedge in $E$, $D$ and $\Omega$ be bounded open sets in $X$, $\theta \in \Omega \subset \overline{\Omega} \subset D$. Suppose that $T : \overline{D} \to X$ is a semiclosed 1-set-contraction mapping, and $x_2 \in X$, $x_1 \in D$ such that

(i) $x \neq tTx + (1 - t)x_1, x \in \partial D, t \in (0, 1].$

(ii) $x \neq Tx + tx_2, x \in \partial \Omega, t \geq 0.$

Then there exists $x_0 \in \overline{D} \setminus \Omega$ such that $x_0 = Tx_0.$

PROOF. If there exists $x_0 \in \partial D$ such that $x_0 = Tx_0$, this theorem is proved.

If for $x \in \partial D$, $x \neq Tx$, then from the proof of Theorem 3 and by Theorem 4 and Theorem 1(2) we obtain

$$i(T, D \setminus \Omega) = i(T, D) - i(T, \Omega) = 1.$$ 

Therefore there exists $x_0 \in D \setminus \overline{\Omega}$ such that $x_0 = Tx_0$. Q.E.D.

COROLLARY. Let $P$ be a cone in real Banach space $X$, and $T : \overline{P}_r \to P$ a semiclosed 1-set-contraction mapping, then there exist $r_1, r_2 : 0 < r_2 < r < r_1 < r$ such that

(i) For all $x \in \partial P_{r_1}$ and $\lambda \geq 1$, $Tx \neq \lambda x$.

(ii) There exists $x_0 \in P$ ($x_0 \neq \emptyset$) such that for all $x \in \partial P_{r_2}$ and $\lambda \geq 0$, $x - Tx \neq \lambda x_0$, where $P_r = \{x \in P ||x|| < r\}$, $\partial P_r = \{x \in P ||x|| = r\}$.

Then $T$ has a fixed point $x^* \in P_{r_1,r_2}$, i.e., $r_2 < ||x^*|| < r_1$.

THEOREM 6. Let $T : P_r \to P$ be a semiclosed 1-set-contraction mapping and $r_1 > r_2 > 0$ ($r_1 < r$) such that the following holds:

(i) $x \in \partial P_{r_1} \Rightarrow Tx \notin x; x \in \partial P_{r_2} \Rightarrow x \notin Tx$,

or

(ii) $x \in \partial P_{r_2} \Rightarrow Tx \notin x; x \in \partial P_{r_1} \Rightarrow x \notin Tx$.

Then there exists a fixed point $x^* \in P_{r_1,r_2}$, i.e., $r_2 < ||x^*|| < r_1$.

PROOF. By the corollary of Theorem 5 we obtain this theorem.

LEMMA. Let $P$ be a cone of real Banach space $E$ and the norm be monotonically increasing with respect to $P$. Suppose that $A : \overline{P}_r \to P$ is a $K$-set-contraction mapping $(0 < K < 1)$ which satisfies the following conditions:

(H1) $x \in \partial P_r \Rightarrow ||Ax|| \leq ||x||; x \in \partial P \Rightarrow ||Ax|| \geq ||x||$,

or

(H2) $x \in \partial P_r \Rightarrow ||Ax|| \leq ||x||; x \in \partial P_r \Rightarrow ||Ax|| \geq ||x||$.

Then $A$ has a fixed point in $\overline{P}_{r,R}$, where $P_{r,R} = P_r \setminus \overline{P}_r$.

PROOF. We prove only that this theorem holds under (H1). Mapping $A$ is a strict-set-contraction. Set $s = \frac{1}{2}(r + R)$.

We set operator $A_n$ as follows:

$$A_n x = \begin{cases} 
(1 + \frac{||x|| - s}{n(R - s)})Ax, & x \in P, s \leq ||x|| \leq R, \\
(1 - \frac{s - ||x||}{n(3 - r)})Ax, & x \in P, r \leq ||x|| < s.
\end{cases}$$

$A_n$ is a continuous and bounded operator.

We discuss operator

$$B_n x = \frac{||x||}{nk}Ax, \quad x \in \overline{P}_{r,R}, \quad k = \min\{R - s, r - s\} = \frac{1}{2}(R - r).$$
Let $\Omega$ be an arbitrary open subset in $\overline{P}_{r,R}$. By definition and property of the measure $\alpha(\Omega)$ of noncompactness of $\Omega$,

$$\alpha(\Omega) = \inf \left\{ d|s_i \subset \Omega, \bigcup_{n=1}^{N} s_i = \Omega, \text{diam } s_i < d \right\},$$

$$\alpha(\Omega) \leq \alpha(\overline{P}_{r,R}), \quad B_n(s_i) \subset B_n(\Omega), \quad \bigcup_{i=1}^{N} B_n(s_i) = B_n(\Omega),$$

$$\text{diam } B_n(s_i) \leq \frac{2R}{nk} \text{diam } A(s_i),$$

$$\alpha(B_n(\Omega)) \leq \frac{2R}{nk} \alpha(A(\Omega)) \leq \frac{2R}{nk} K \alpha(\Omega).$$

Then $B_n$ is a $(2R/nk)K$-set-contraction mapping. As $n \to \infty$, $(2R/nk)K \to 0$, i.e., when $n$ is large enough, $B_n$ is a strict-set-contraction mapping on $\overline{P}_{r,R}$.

In addition, we set operators:

$$A_n^{(1)} x = \left( 1 + \frac{\|x\| - s}{n(R-s)} \right) Ax, \quad x \in P, \quad s \leq \|x\| \leq R,$$

$$A_n^{(2)} x = \left( 1 - \frac{s - \|x\|}{n(s-r)} \right) Ax, \quad x \in P, \quad r \leq \|x\| < s.$$

Obviously, $A_n^{(1)}$ and $A_n^{(2)}$ are $\hat{k}$-set-contraction mappings, where $\hat{k} \leq (1 + (2R + s/nk))K$. When $n$ is large enough, $0 < \hat{k} < 1$. Hence $A_n^{(1)}$ and $A_n^{(2)}$ are strict-set-contraction mappings for $n$ large enough. Set

$$P_1 = \{ x \in P | s \leq \|x\| \leq R \}, \quad P_2 = \{ x \in P | r \leq \|x\| < s \}, \quad \overline{P}_{r,R} = P_1 \cup P_2.$$

Hence, $\Omega = (\Omega \cap P_1) \cup (\Omega \cap P_2)$,

$$A_n(\Omega) = [A_n(\Omega \cap P_1)] \cup [A_n(\Omega \cap P_2)],$$

$$= [A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)],$$

$$\alpha(A_n(\Omega)) = \alpha([A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)]),$$

$$= \max\{\alpha(A_n^{(1)}(\Omega \cap P_1)), \alpha(A_n^{(2)}(\Omega \cap P))\},$$

$$\leq \hat{k} \alpha(\Omega), \quad 0 < \hat{k} < 1.$$

Then $A_n$ is a strict-set-contraction mapping and when $n$ is large enough, $A_n$ satisfies the conditions of Theorem 6.

In fact, if there is $x_0 \in \partial P_r$ such that $A_n x_0 \geq x_0$, then the norm $\|x\|$ is monotonically increasing and by condition (H1),

$$r = \|x_0\| \leq \|A_n x_0\| = (1 - 1/n)\|Ax_0\| \leq (1 - 1/n)\|x_0\| = (1 - 1/n)r < r.$$

This is a contradiction. Then $x \in \partial P_r$, $A_n x \notin x$.

If there is $x_1 \in \partial P_R$ such that $A_n x_1 \leq x_1$, we obtain the following:

$$R = \|x_1\| \geq \|A_n x_1\| = (1 + 1/n)\|Ax_1\| \geq (1 + 1/n)\|x_1\| = (1 + 1/n)R > R.$$

This is also a contradiction. Then $x \in \partial P_R$, $A_n x \notin x$. Hence $A_n$ satisfies the conditions of Theorem 6.
There is $x^* \in \overline{P}_{r,R}$ such that $A_n x^* = x^*$. Without loss of generality, suppose that $P_1$ includes subsequence $\{x^n_k\}$ of $\{x^n\}$,

\begin{align}
    x^n_k = A_n x^n_k = \left(1 + \frac{\|x^n_k\| - s}{n_k(R - s)}\right) A x^n_k.
\end{align}

Since $A$ is a strict-set-contraction mapping, the set $\{|A x^n_k|\}$ must be bounded:

\begin{align}
    \|x^n_k - A x^n_k\| = \frac{\|x^n_k\| - s}{n_k(R - s)} \|A x^n_k\| < \frac{1}{n_k} \|A x^n_k\| \rightarrow 0 \quad (n_k \rightarrow \infty),
\end{align}

i.e., $x^n_k - A x^n_k \rightarrow \theta$ $(n_k \rightarrow \infty)$.

Since a strict-set-contraction mapping is a semicompact 1-set-contraction, then there exists a convergent subsequence of $\{x^n_k\}$, which we write down also as $\{x^n_k\}$, and $x^n_k \rightarrow x^*$ $(n_k \rightarrow \infty)$. Since $\overline{P}_{r,R}$ is closed, $x^* \in \overline{P}_{r,R}$. By (4), as $n_k \rightarrow \infty$, we imply that $A x^* = x^*$. This proves that $A$ has a fixed point $x^* \in \overline{P}_{r,R}$ under $(H_1)$.

Analogously, we can prove that $A$ has a fixed point $x^* \in \overline{P}_{r,R}$ under $(H_2)$.

**Theorem 7.** Let $P$ be a cone of a real Banach space $E$ and let the norm be monotone with respect to $P$. Suppose that $A: \overline{P}_{r,R} \rightarrow P$ is a semiclosed 1-set-contraction mapping and there is $\delta > 0$ such that

(i) $x \in \partial P_r \Rightarrow \|Ax\| < \|x\|$; $x \in \partial P_R \Rightarrow \|Ax\| \geq (1 + \delta)\|x\|,$

or

(ii) $x \in \partial P_R \Rightarrow \|Ax\| \leq \|x\|$; $x \in \partial P_R \Rightarrow \|Ax\| \geq (1 + \delta)\|x\|.$

Then $A$ has a fixed point in $\overline{P}_{r,R}$.

**Proof.** (i) Since $A: \overline{P}_{r,R} \rightarrow P$ is a semiclosed 1-set-contraction mapping, then for an arbitrary open subset $\Omega \subset \overline{P}_{r,R}$,

\begin{align}
    \alpha(A(\Omega)) \leq \alpha(\Omega).
\end{align}

We construct operator $A_n$ as follows:

\begin{align}
    A_n x = \lambda_n A, \quad \lambda_n = \frac{n - 1}{n}.
\end{align}

Hence $\alpha(A_n(\Omega)) = \lambda_n \alpha(A(\Omega)) \leq \lambda_n \alpha(\Omega)$. Hence $A_n$ is a strict-set-contraction mapping.

If the condition (i) holds, when $n$ is large enough that $1 > \lambda_n > 1/(1 + \delta)$, then

for $x \in \partial P_r$ \quad $\|A_n x\| = \lambda_n \|Ax\| < \|Ax\| \leq \|x\|;$

for $x \in \partial P_R$ \quad $\|A_n x\| = \lambda_n \|Ax\| = \lambda_n \|Ax\| > \frac{1}{1 + \delta} \|Ax\| \geq \|x\|.$

$A_n$ satisfies the conditions of the lemma, hence there exists $x_n \in \overline{P}_{r,R}$, such that

\begin{align}
    A_n x_n = x_n, \quad \text{i.e.,} \lambda_n A x_n = x_n, \\
    x_n - A x_n = x_n - \lambda_n A x_n + \lambda_n A x_n - A x_n = (\lambda_n - 1) A x_n \rightarrow \theta \quad (n \rightarrow \infty).
\end{align}

Because $(I - A)$ is closed and $\overline{P}_{r,R}$ is a closed set, there is $x_0 \in \overline{P}_{r,R}$ such that $x_0 = A x_0$.

Analogously we can prove that $A$ has a fixed point $x^* \in \overline{P}_{r,R}$ under condition (ii). Q.E.D.
THEOREM 8. Let $E$ be a real Banach space and $P$ be a normal cone of $E$. Suppose that $T: [0, y] \to E$ is a monotonically increasing semiclosed 1-set-contraction mapping and $T(0) \geq 0$, $T(y) \leq y$. Then $T$ has a fixed point in $[0, y]$.

PROOF. We set $T_m x = t_m T x$, where $m$ is an arbitrary natural number and $t_m = (m - 1)/m$. Obviously, $T_m: [0, y] \to E$ is a monotonically increasing $t_m$-set-contraction mapping $(0 < t_m < 1)$. We have

$$T_m(0) = t_m T(0) \geq t_m \cdot 0 \geq 0; \quad T_m(y) = t_m T(y) \leq t_m y \leq y;$$

hence $T_m$ satisfies the conditions of Theorem 3 of [3], it has a minimal fixed point $\tilde{x}_m$ and

$$\tilde{x}_m = \lim_{n \to \infty} T_m^{(n)}(0),$$

$$\tilde{x}_m - T(\tilde{x}_m) = T_m(\tilde{x}_m) - T(\tilde{x}_m) = (t_m - 1)T(\tilde{x}_m) \to \theta \quad (m \to \infty).$$

Since $(I - T)$ is closed, $\theta \in (I - T)[0,y]$. Then there is $\bar{x} \in [0,y]$ such that $\bar{x} - T\bar{x} = \theta$, i.e., $\bar{x} = T\bar{x}$. Q.E.D.

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REFERENCES


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