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THE FIXED-POINT INDEX AND THE FIXED-POINT THEOREMS OF 1-SET-CONTRACTION MAPPINGS

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ABSTRACT. W. V. Petryshyn [1] studied the fixed-point theorem of 1-set-contraction mappings. This paper gives the definition of the fixed-point index of 1-set-contraction mappings and the concept of semiclosed 1-set-contraction mappings; then we obtain some fixed-point theorems about it.

In this paper we define the fixed-point index of 1-set-contraction mappings and give the concept of semiclosed 1-set-contraction mappings. Hence the fixed-point index of strict-set-contraction mappings (Nussbaum [**2**]) is extended to 1-set-contraction mappings. We obtain furthermore some fixed-point theorems of semiclosed 1-set-contraction mappings. Let E be a real Banach space, X a nonempty closed convex subset of E, D a relative bounded open set with respect to X, and $\overline{D}$ and ∂D the closure and boundary of D in X respectively.

Suppose that $T\colon \overline{D} \to X$ is 1-set-contraction mapping and $\theta \notin \overline{(I-T)\partial D}$, so there exists $\delta > 0$ such that

$$\inf_{x\in\partial D} \|x - Tx\| \geq \delta. \tag{1}$$

We set $T_k = kT$, where $k \in (1-\delta/M, 1)$, $M = \sup_{x\in\overline{D}} \|Tx\| + \delta$. Obviously T_k is a strict-set-contraction mapping. By Amann [**3**], T_k has fixed-point index $i(T_k, D)$.

DEFINITION 1. The fixed-point index $i(T, D)$ of T on D is defined by $i(T, D) = i(T_k, D)$.

We can prove that $i(T, D)$ is defined uniquely and independent of T_k. In fact, suppose that $W_i\colon \overline{D} \to X$ is a K_i-set-contraction mapping $(0 < K_i < 1)$ with

$$\|W_i x - Tx\| < \delta \qquad x \in \partial D,\ i = 1, 2. \tag{2}$$

We make a homotopic mapping on $\overline{D}$ as follows

$$H(t, x) = tW_1 x + (1-t)W_2 x \qquad x \in \overline{D},\ t \in [0, 1].$$

$H_t\colon \overline{D} \to X$ is a λ-set-contraction mapping, where $\lambda = \max\{k_1, k_2\}$. For every $x \in \partial D$

$$\begin{aligned}\|x - H(t, x)\| &= \|x - tW_1 x - (1-t)W_2 x\| \\ &\geq \|x - Tx\| - t\|Tx - W_1 x\| - (1-t)\|Tx - W_2 x\| \\ &> \delta - t\delta - (1-t)\delta = 0.\end{aligned}$$

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By Proposition 2.1(3) of [**3**], $i(H(t,x),D) = \text{const}$. Then

$$i(H(1,x),D) = i(H(0,x),D),$$

i.e., $i(W_1,D) = i(W_2,D)$. This equality shows that $i(T,D)$ is independent of T_k. $i(T,D)$ satisfies three axioms of the fixed-point index:

THEOREM 1. (I) *Let* $Tx = x_0$ $\forall x \in \overline{D}$;

$$i(T,D) = \begin{cases} 1, & x_0 \in D, \\ 0, & x_0 \notin D. \end{cases}$$

(II) Suppose that D, D_1, and D_2 are open subsets of X, $D_1 \cup D_2 \subset D$ and $D_1 \cap D_2 = \varnothing$, T has no fixed-point on $\overline{D} \setminus (D_1 \cup D_2)$. Then $i(T,D) = i(T,D_1) + i(T,D_2)$.

(III) Suppose that the mapping $H\colon [0,1] \times \overline{D} \to X$ is continuous and $\theta \notin (I - H(t,x))([0,1] \times \partial D)$. Suppose that for every $t \in [0,1]$, $H(t,\cdot)\colon \overline{D} \to X$ is a 1-set-contraction mapping. Finally suppose that $H(\cdot,x)\colon [0,1] \to X$ is uniformly continuous with respect to $x \in \overline{D}$. Then $i(H(1,x),D) = i(H(0,x),D)$ and $i(H(t,x),D)$ is independent of t.

PROOF. (I) Since T is a constant mapping, T must be a 0-set-contraction mapping. Hence

$$i(T,D) = \begin{cases} 1, & x_0 \in D, \\ 0, & x_0 \notin D. \end{cases}$$

(II) As discussion in Definition 1, we set

$$\delta = \inf\{\|x - Tx\| \mid x \in \overline{D} \setminus (D_1 \cup D_2)\}.$$

Suppose that $W\colon \overline{D} \to X$ is a strict-set-contraction mapping such that

$$\|Tx - Wx\| < \delta, \qquad x \in \overline{D} \setminus (D_1 \cup D_2). \tag{3}$$

By Definition 1,

$$i(T,D) = i(W,D) \quad \text{and} \quad (T,D_j) = i(W,D_j) \qquad (j = 1,2).$$

W has no fixed point on $\overline{D} \setminus (D_1 \cup D_2)$.

In fact, if there is $x_0 \in \overline{D} \setminus (D_1 \cup D_2)$ such that $x_0 = Wx_0$, then

$$\delta \le \|x_0 - Tx_0\| = \|x_0 - Wx_0 + Wx_0 - Tx_0\| = \|Wx_0 - Tx_0\|,$$

which contradicts (3).

By Proposition 2.1 of [**3**], $i(W,D) = i(W,D_1) + i(W,D_2) = i(T,D_1) + i(T,D_2)$; then $i(T,D) = i(T,D_1) + i(T,D_2)$.

(III) By the definition of $H(t,x)$, there are $\delta > 0$ and $M > 0$ such that

$$\|x - H(t,x)\| \ge \delta > 0 \qquad t \in [0,1],\ x \in \partial D,$$

and $\|H(t,x)\| < M$, $t \in [0,1]$, $x \in \overline{D}$.

Set $k \in (1 - \delta/2M, 1)$,

$$G(t,x) = kH(t,x)\colon [0,1] \times \overline{D} \to X,$$
$$\|H(t,x) - G(t,x)\| \le (1-k)M < \delta/2, \qquad t \in [0,1],\ x \in \overline{D};$$

then

$$\|G(t,x) - x\| \ge \|x - H(t,x)\| - \|H(t,x) - G(t,x)\| > \delta - \delta/2 > 0.$$

For every $t \in [0,1]$, obviously $G(t,x)\colon \overline{D} \to X$ is a k-set-contraction and $G(\cdot,x)\colon [0,1] \to X$ is uniformly continuous with respect to $x \in \overline{D}$.

By Proposition 2.1(3) of [**3**],

$$i(G(t,x),D) = \text{const}. \qquad (t \in [0,1]).$$

According to Definition 1

$$i(H(t,x),D) = i(G(t,x),D),$$

then

$$i(H(1,x),x) = i(H(0,x),D). \quad \text{Q.E.D.}$$

DEFINITION 2. T is called a semiclosed 1-set-contraction mapping, if T is 1-set-contraction and $I - T$ is closed.

Let us state a definition of semicompact mapping [**6**].

$T\colon E \to E$ is said to be semicompact if for each bounded sequence $\{x_n\}$ in E such that $x_n - T(x_n) \to y$ for some y in E, there exists a convergent subsequence.

Obviously, semicompact 1-set-contraction mapping $\not\Rightarrow$ semiclosed 1-set-contraction mapping.

THEOREM 2 (SOLVABILITY). *Let E be a real Banach space, X be a closed convex subset of E, D be a bounded open set with respect to X.*

Suppose that $T\colon \overline{D} \to X$ is semiclosed 1-set-contraction mapping, $\theta \notin (I-T)\partial D$, $i(T,D) \neq 0$. Then T has a fixed point in D.

PROOF. Suppose that $k_n \in (0,1)$ and $k_n \to 1$. We set $W_n = k_n T$, $W_n\colon \overline{D} \to X$ is k_n-set-contraction. Since $\operatorname{Sup}_{x\in\overline{D}} \|Tx\| < +\infty$,

$$\|Tx - W_n x\| = \|Tx - k_n Tx\| = (1-k_n)\|Tx\| \to 0 \qquad (n \to \infty),$$

then there is N, such that for every $n > N$

$$\|Tx - W_n x\| < \delta \quad \text{where } 0 < \delta < \inf_{x\in\partial D} \|x - Tx\|.$$

By the definition and our assumptions, $i(T,D) = i(W_n,D) \neq 0$. By Proposition 2.1(1) of [**3**], W_n has a fixed point $x_n \in D$, i.e., $x_n = W_n x_n$. Hence for every $n > N$

$$\|x_n - Tx_n\| = \|x_n - W_n x_n - Tx_n + W_n x_n\| = \|W_n x_n - Tx_n\| \to 0 \qquad (n \to \infty).$$

Then $x_n - Tx_n \to \theta$ $(n \to \infty)$.

Since $I - T$ is closed, so $\theta \in (I-T)\overline{D}$, there is $x_0 \in \overline{D}$ such that $x_0 = Tx_0$. Because $\theta \notin (I-T)\partial D$, there exists $x_0 \in D$ such that $x_0 = Tx_0$. Q.E.D.

THEOREM 3. *Let E be a real Banach space, X be a closed convex set in E, D be a bounded open set in X. Suppose that $T\colon \overline{D} \to X$ is a semiclosed 1-set-contraction mapping, and that $x_0 \in D$ is such that*

(a) $$x \neq tTx + (1-t)x_0, \qquad x \in \partial D,\ t \in (0,1].$$

Then T has a fixed point in $\overline{D}$.

PROOF. If T has a fixed point on ∂D, the conclusion of this theorem obviously holds.

If T has no fixed point on ∂D, then $i(T, D)$ has meaning and

$$x \neq tTx + (1-t)x_0 \qquad x \in \partial D,\ t \in [0,1].$$

Set

$$H(t,x) = tTx + (1-t)x_0.$$

We know easily that $H(t,x)$ satisfies the conditions of Theorem 1(III), so that

$$i(T,D) = i(x_0, D) = 1.$$

By Theorem 2, T has a fixed point in D. Q.E.D.

If $T(\partial D) \subset D$, $x_0 \in D$, for arbitrary $x \in \partial D$, $0 < t \leq 1$, we have that

$$tTx + (1-t)x_0 \neq x, \qquad x \in \partial D.$$

Then the condition (a) is satisfied. We have the following consequences:

COROLLARY 1. *If $T(\partial D) \subset D$, then T has a fixed point in D, where D is convex.*

COROLLARY 2. *If $x_0 = \theta \in D$ and $x \neq Tx$, $x \in \partial D$, $t \in (0,1]$, then $i(T,D) = 1$, and T has a fixed point in D.*

THEOREM 4. *Let E be a real Banach space, X be a wedge in E (Amann [**5**]), and D be a bounded open set in X. Suppose that $T\colon \overline{D} \to X$ is a semiclosed 1-set-contraction mapping and $x_0 \in X$, $x_0 \neq \theta$ such that*

$$x \neq Tx + \lambda x_0, \qquad \lambda \geq 0,\ x \in \partial D,$$

then $i(T,D) = 0$.

PROOF. Since $\overline{D}$ is bounded, there exists $\lambda_0 > 0$ such that $\forall \lambda \geq \lambda_0$, $\lambda x_0 \notin \overline{D}$. We set

$$H_\lambda(t,x) = (1-t)Tx + t\lambda x_0, \qquad \lambda \geq \lambda_0,\ t \in [0,1],\ x \in \overline{D}.$$

For arbitrary $t \in [0,1]$ and $\lambda \geq \lambda_0$, $H_\lambda(t,\cdot)\colon \overline{D} \to X$ is a 1-set-contraction mapping.

We can prove that $H_\lambda(t,x) \neq x$, where $x \in \partial D$, $t \in [0,1]$, $\lambda \geq \lambda_1$ for some $\lambda_1 \geq \lambda_0$, in fact, if the present inequality does not hold, then there are $\lambda_n \to \infty$ and corresponding $x_n \in \partial D$ and $t_n \in [0,1]$ such that

$$x_n = (1-t_n)Tx_n + t_n\lambda_n x_0.$$

Since the set $\{x_n - (1-t_n)Tx_n\}$ is bounded (and hence $\{t_n\lambda_n\}$ is bounded), we may suppose that $\lambda_n t_n \to l_0 < +\infty$; hence $t_n \to 0$ $(n \to \infty)$, and

$$x_n - Tx_n = [x_n - (1-t_n)Tx_n] - t_nTx_n \to l_0x_0.$$

Because $I - T$ is closed, $l_0x_0 \in (I-T)\partial D$. Then there is $x^* \in \partial D$ such that

$$(I-T)x^* = l_0x_0,$$

which contradicts our assumptions. By Theorem 1 and $\lambda x_0 \notin \overline{D}$ $(\lambda \geq \lambda_0)$, we obtain

$$i(T,D) = i(\lambda x_0, D) = 0. \quad \text{Q.E.D.}$$

THEOREM 5. *Let E be a real Banach space, X be a wedge in E, D and Ω be bounded open sets in X, $\theta \in \Omega \subset \overline{\Omega} \subset D$. Suppose that $T\colon \overline{D} \to X$ is a semiclosed 1-set-contraction mapping, and $x_2 \in X$, $x_1 \in D$ such that*

(i) $x \neq tTx + (1-t)x_1$, $x \in \partial D$, $t \in (0,1]$.

(ii) $x \neq Tx + tx_2$, $x \in \partial\Omega$, $t \geq 0$.

Then there exists $x_0 \in \overline{D} \setminus \Omega$ such that $x_0 = Tx_0$.

PROOF. If there exists $x_0 \in \partial D$ such that $x_0 = Tx_0$, this theorem is proved.

If for $x \in \partial D$, $x \neq Tx$, then from the proof of Theorem 3 and by Theorem 4 and Theorem 1(II) we obtain

$$i(T, D \setminus \overline{\Omega}) = i(T, D) - i(T, \Omega) = 1.$$

Therefore there exists $x_0 \in D \setminus \overline{\Omega}$ such that $x_0 = Tx_0$. Q.E.D.

COROLLARY. *Let P be a cone in real Banach space X, and $T\colon \overline{P}_r \to P$ a semiclosed 1-set-contraction mapping, then there exist r_1, $r_2\colon 0 < r_2 < r_1 < r$ such that*

(i) *For all $x \in \partial P_{r_1}$ and $\lambda \geq 1$, $Tx \neq \lambda x$.*

(ii) *There exists $x_0 \in P$ $(x_0 \neq \theta)$ such that for all $x \in \partial P_{r_2}$ and $\lambda \geq 0$, $x - Tx \neq \lambda x_0$, where $P_r = \{x \in P \mid \|x\| < r\}$, $\partial P_r = \{x \in P \mid \|x\| = r\}$.*

Then T has a fixed point $x^* \in P_{r_1,r_2}$, i.e., $r_2 < \|x^*\| < r_1$.

THEOREM 6. *Let $T\colon P_r \to P$ be a semiclosed 1-set-contraction mapping and $r_1 > r_2 > 0$ $(r_1 < r)$ such that the following holds:*

(i) $x \in \partial P_{r_1} \Rightarrow Tx \ngeq x$; $x \in \partial P_{r_2} \Rightarrow x \ngeq Tx$,

or

(ii) $x \in \partial P_{r_2} \Rightarrow Tx \ngeq x$; $x \in \partial P_{r_1} \Rightarrow x \ngeq Tx$.

Then there exists a fixed point $x^* \in P_{r_1,r_2}$, i.e., $r_2 < \|x^*\| < r_1$.

PROOF. By the corollary of Theorem 5 we obtain this theorem.

LEMMA. *Let P be a cone of real Banach space E and the norm be monotonically increasing with respect to P. Suppose that $A\colon \overline{P}_{r,R} \to P$ is a K-set-contraction mapping $(0 < K < 1)$ which satisfies the following conditions:*

(H_1) $x \in \partial P_r \Rightarrow \|Ax\| \leq \|x\|$; $x \in \partial P_R \Rightarrow \|Ax\| \geq \|x\|$,

or

(H_2) $x \in \partial P_R \Rightarrow \|Ax\| \leq \|x\|$; $x \in \partial P_r \Rightarrow \|Ax\| \geq \|x\|$.

Then A has a fixed point in $\overline{P}_{r,R}$, where $P_{r,R} = P_R \setminus \overline{P}_r$.

PROOF. We prove only that this theorem holds under (H_1). Mapping A is a strict-set-contraction. Set $s = \frac{1}{2}(r+R)$.

We set operator A_n as follows:

$$A_n x = \begin{cases} (1 + \frac{\|x\| - s}{n(R-s)})Ax, & x \in P,\ s \leq \|x\| \leq R, \\ (1 - \frac{s - \|x\|}{n(s-r)})Ax, & x \in P,\ r \leq \|x\| < s. \end{cases}$$

A_n is a continuous and bounded operator.

We discuss operator

$$B_n x = \frac{\|x\|}{nk} Ax, \qquad x \in \overline{P}_{r,R},\ k = \min\{R-s, r-s\} = \frac{1}{2}(R-r).$$

Let Ω be an arbitrary open subset in $\overline{P}_{r,R}$. By definition and property of the measure $\alpha(\Omega)$ of noncompactness of Ω,

$$\alpha(\Omega) = \inf\left\{d | s_i \subset \Omega, \bigcup_{n=1}^{N} s_i = \Omega, \operatorname{diam} s_i < d\right\},$$

$$\alpha(\Omega) \le \alpha\{\overline{P}_{r,R}\}, \quad B_n(s_i) \subset B_n(\Omega), \quad \bigcup_{i=1}^{N} B_n(s_i) = B_n(\Omega),$$

$$\operatorname{diam} B_n(s_i) \le \frac{2R}{nk} \operatorname{diam} A(s_i),$$

$$\alpha(B_n(\Omega)) \le \frac{2R}{nk}\alpha(A(\Omega)) \le \frac{2R}{nk} K\alpha(\Omega).$$

Then B_n is a $(2R/nk)K$-set-contraction mapping. As $n \to \infty$, $(2R/nk)K \to 0$, i.e., when n is large enough, B_n is a strict-set-contraction mapping on $\overline{P}_{r,R}$.

In addition, we set operators:

$$A_n^{(1)}x = \left(1 + \frac{\|x\| - s}{n(R-s)}\right) Ax, \qquad x \in P,\ s \le \|x\| \le R,$$

$$A_n^{(2)}x = \left(1 - \frac{s - \|x\|}{n(s-r)}\right) Ax, \qquad x \in P,\ r \le \|x\| < s.$$

Obviously, $A_n^{(1)}$ and $A_n^{(2)}$ are $\tilde{k}$-set-contraction mappings, where $\tilde{k} \le (1 + (2R + s/nk))K$. When n is large enough, $0 < \tilde{k} < 1$. Hence $A_n^{(1)}$ and $A_n^{(2)}$ are strict-set-contraction mappings for n large enough. Set

$$P_1 = \{x \in P | s \le \|x\| \le R\},\ P_2 = \{x \in P | r \le \|x\| < s\}, \qquad \overline{P}_{r,R} = P_1 \cup P_2.$$

Hence, $\Omega = (\Omega \cap P_1) \cup (\Omega \cap P_2)$,

$$\begin{aligned} A_n(\Omega) &= [A_n(\Omega \cap P_1)] \cup [A_n(\Omega \cap P_2)], \\ &= [A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)], \end{aligned}$$

$$\begin{aligned} \alpha(A_n(\Omega) &= \alpha\{[A_n^{(1)}(\Omega \cap P_1)] \cup [A_n^{(2)}(\Omega \cap P_2)]\}, \\ &= \operatorname{Max}\{\alpha[A_n^{(1)}(\Omega \cap P_1)], \alpha[A_n^{(2)}(\Omega \cap P)]\}, \\ &\le \tilde{k}\alpha(\Omega), \qquad 0 < \tilde{k} < 1. \end{aligned}$$

Then A_n is a strict-set-contraction mapping and when n is large enough, A_n satisfies the conditions of Theorem 6.

In fact, if there is $x_0 \in \partial P_r$ such that $A_n x_0 \ge x_0$, then the norm $\|x\|$ is monotonically increasing and by condition (H_1),

$$r = \|x_0\| \le \|A_n x_0\| = (1 - 1/n)\|Ax_0\| \le (1 - 1/n)\|x_0\| = (1 - 1/n)r < r.$$

This is a contradiction. Then $x \in \partial P_r$, $A_n x \ngeq x$.

If there is $x_1 \in \partial P_R$ such that $A_n x_1 \le x_1$, we obtain the following:

$$R = \|x_1\| \ge \|A_n x_1\| = (1 + 1/n)\|Ax_1\| \ge (1 + 1/n)\|x_1\| = (1 + 1/n)R > R.$$

This is also a contradiction. Then $x \in \partial P_R$, $A_n x \nleq x$. Hence A_n satisfies the conditions of Theorem 6.

There is $x^* \in \overline{P}_{r,R}$ such that $A_n x^* = x^*$. Without loss of generality, suppose that P_1 includes subsequence $\{x^*_{n_k}\}$ of $\{x^*_n\}$,

$$x^*_{n_k} = A_{n_k} x^*_{n_k} = \left(1 + \frac{\|x^*_{n_k}\| - s}{n_k(R-s)}\right) A x^*_{n_k}. \tag{4}$$

Since A is a strict-set-contraction mapping, the set $\{\|Ax^*_{n_k}\|\}$ must be bounded:

$$\|x^*_{n_k} - Ax^*_{n_k}\| = \frac{\|x^*_{n_k}\| - s}{n_k(R-s)}\|Ax^*_{n_k}\| < \frac{1}{n_k}\|Ax^*_{n_k}\| \to 0 \qquad (n_k \to \infty),$$

i.e., $x^*_{n_k} - Ax^*_{n_k} \to \theta$ $(n_k \to \infty)$.

Since a strict-set-contraction mapping is a semicompact 1-set-contraction, then there exists a convergent subsequence of $\{x^*_{n_k}\}$, which we write down also as $\{x^*_{n_k}\}$, and $x^*_{n_k} \to x^*$ $(n_k \to \infty)$. Since $\overline{P}_{r,R}$ is closed, $x^* \in \overline{P}_{r,R}$. By (4), as $n_k \to \infty$, we imply that $Ax^* = x^*$. This proves that A has a fixed point $x^* \in \overline{P}_{r,R}$ under (H_1). Analogously, we can prove that A has a fixed point $x^* \in \overline{P}_{r,R}$ under (H_2).

THEOREM 7. *Let P be a cone of a real Banach space E and let the norm be monotone with respect to P. Suppose that $A\colon \overline{P}_{r,R} \to P$ is a semiclosed 1-set-contraction mapping and there is $\delta > 0$ such that*

(i) $x \in \partial P_r \Rightarrow \|Ax\| \le \|x\|$; $x \in \partial P_R \Rightarrow \|Ax\| \ge (1+\delta)\|x\|$,

or

(ii) $x \in \partial P_R \Rightarrow \|Ax\| \le \|x\|$; $x \in \partial P_r \Rightarrow \|Ax\| \ge (1+\delta)\|x\|$.

Then A has a fixed point in $\overline{P}_{r,R}$.

PROOF. (i) Since $A\colon \overline{P}_{r,R} \to P$ is a semiclosed 1-set-contraction mapping, then for an arbitrary open subset $\Omega \subset \overline{P}_{r,R}$,

$$\alpha(A(\Omega)) \le \alpha(\Omega).$$

We construct operator A_n as follows:

$$A_n x = \lambda_n A, \ \lambda_n = \frac{n-1}{n}.$$

Hence $\alpha(A_n(\Omega)) = \alpha(\lambda_n(A(\Omega)) = \lambda_n \alpha(A(\Omega)) \le \lambda_n \alpha(\Omega)$. Hence A_n is a strict-set-contraction mapping.

If the condition (i) holds, when n is large enough that $1 > \lambda_n > 1/(1+\delta)$, then

$$\text{for } x \in \partial P_r \qquad \|A_n x\| = \lambda_n \|Ax\| < \|Ax\| \le \|x\|;$$

$$\text{for } x \in \partial P_R \qquad \|A_n x\| = \|\lambda_n Ax\| = \lambda_n \|Ax\| > \frac{1}{1+\delta}\|Ax\| \ge \|x\|.$$

A_n satisfies the conditions of the lemma, hence there exists $x_n \in \overline{P}_{r,R}$, such that

$$A_n x_n = x_n, \quad \text{i.e., } \lambda_n A x_n = x_n,$$

$$x_n - Ax_n = x_n - \lambda_n A x_n + \lambda_n A x_n - A x_n = (\lambda_n - 1) A x_n \to \theta \qquad (n \to \infty).$$

Because $(I - A)$ is closed and $\overline{P}_{r,R}$ is a closed set, there is $x_0 \in \overline{P}_{r,R}$ such that $x_0 = Ax_0$.

Analogously we can prove that A has a fixed point $x^* \in \overline{P}_{r,R}$ under condition (ii). Q.E.D.

THEOREM 8. *Let E be a real Banach space and P be a normal cone of E. Suppose that $T\colon [0,y] \to E$ is a monotonically increasing semiclosed 1-set-contraction mapping and $T(0) \geq 0$, $T(y) \leq y$. Then T has a fixed point in $[0,y]$.*

PROOF. We set $T_m x = t_m T x$, where m is an arbitrary natural number and $t_m = (m-1)/m$. Obviously, $T_m\colon [0,y] \to E$ is a monotonically increasing t_m-set-contraction mapping $(0 < t_m < 1)$. We have

$$T_m(0) = t_m T(0) \geq t_m \cdot 0 \geq 0; \qquad T_m(y) = t_m T(y) \leq t_m y \leq y;$$

hence T_m satisfies the conditions of Theorem 3 of [**3**], it has a minimal fixed point $\tilde{x}_m$ and

$$\tilde{x}_m = \lim_{n\to\infty} T_m^{(n)}(0),$$
$$\tilde{x}_m - T(\tilde{x}_m) = T_m(\tilde{x}_m) - T(\tilde{x}_m) = (t_m - 1)T(\tilde{x}_m) \to \theta \quad (m \to \infty).$$

Since $(I - T)$ is closed, $\theta \in (I-T)[0,y]$. Then there is $\overline{x} \in [0,y]$ such that $\overline{x} - T\overline{x} = \theta$, i.e., $\overline{x} = T\overline{x}$. Q.E.D.

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REFERENCES

1a. W. V. Petryshyn, Trans. Amer. Math. Soc. **182** (1973), 323–352.
1b. ——, Bull. Amer. Math. Soc. **78** (1972), 225–229.
2. D. D. Nussbaum, Ann. Math. Pura. Appl. **89** (1971), 217–258.
3. H. Amann, J. Funct. Anal. **11** (1972), 346–384.
4. N. P. Cac and J. A. Gatica, J. Math. Anal. Appl. **71** (1979), 547–557.
5. H. Amann, SIAM Rev. (1976), 632.
6. W. V. Petryshyn, J. Math. Anal. and Appl. 39 (1972), 717–741.
7. Zhang Qing Yong, Chinese Ann. of Math. (2) **5** (1984), 91–98.

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