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INEQUALITIES FOR α-OPTIMAL PARTITIONING OF A MEASURABLE SPACE

JERZY LEGUT

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ABSTRACT. An α-optimal partition $\{A_i^*\}_{i=1}^n$ of a measurable space according to n nonatomic probability measures $\{\mu_i\}_{i=1}^n$ is defined. A two-sided inequality for $v^* = \min \alpha_i^{-1}\mu_i(A_i^*)$ is given. This estimation generalizes and improves a result of Elton et al. [**3**].

1. Introduction. Let $(\mathscr{X}, \mathscr{B})$ denote a measurable space and let $\{\mu_i\}_{i=1}^n$ be nonatomic probability measures defined on the same σ-algebra $\mathscr{B}$. Let $\mathscr{P}$ stand for the set of all measurable partitions $P = \{A_i\}_{i=1}^n$ of $\mathscr{X}(\bigcup_{i=1}^n A_i = \mathscr{X}$, $A_i \cap A_j = \emptyset$, for all $i \neq j)$.

DEFINITION. A partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}$ is said to be an α-optimal if

$$\min_{i\in I}\{\alpha_i^{-1}\mu_i(A_i^*)\} = \sup\left\{\min_{i\in I}\{\alpha_i^{-1}\mu_i(A_i)\} : P = \{A_i\}_{i=1}^n \in \mathscr{P}\right\},$$

where $I = \{1, 2, \ldots, n\}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$ is a vector satisfying $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i > 0$ for all $i \in I$.

The main purpose of this paper is to give as good as possible estimation of the number $v^* = \min_{i\in I}\{\alpha_i^{-1}\mu_i(A_i^*)\}$ by suitable inequalities. Using other simpler methods we generalize and improve the result of Elton et al. [**3**]. In §2 we state and prove the main theorem.

2. The main result. The problem of the α-optimal partitioning of a measurable space $(\mathscr{X}, \mathscr{B})$ can be interpreted as the well-known problem of fair division (cf. [**1, 6**]) of an object $\mathscr{X}$ with unequal weights. Here, each μ_i, $i \in I$, represents the individual evaluation of sets from $\mathscr{B}$. We also assume in this problem that $\{\mu_i\}_{i=1}^n$ are nonatomic probability measures. Dividing the object $\mathscr{X}$ fairly we are interested in giving the ith person a set $A_i \in \mathscr{B}$ such that $\mu_i(A_i) \geq \alpha_i$, for all $i \in I$.

Under assumptions given above, Dubins and Spanier [**1**] proved the following

THEOREM 1. *Assume that $\mu_i \neq \mu_j$ for some $i \neq j$. Then there exists a partition $P = \{A_i\}_{i=1}^n \in \mathscr{P}$ such that $\mu_i(A_i) > \alpha_i$ for all $i \in I$.*

The proof of Theorem 1 can be derived from the following result of Dvoretzky et al. [**2**] (cf. [**1**]).

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THEOREM 2. *Let $\bar{\mu}\colon \mathscr{P} \to R^n$ denote the division vector valued function defined by*

$$\bar{\mu}(P) = (\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n)) \in R^n, \qquad P = \{A_i\}_{i=1}^n \in \mathscr{P}.$$

Then the range $\bar{\mu}(\mathscr{P})$ of $\bar{\mu}$ is convex and compact in R^n.

From Theorem 2 we conclude

COROLLARY 1. *There exists a partition $P^* = \{A_i^*\}_{i=1}^n \in \mathscr{P}$ such that*

$$v^* = \min_{i\in I}\{\alpha_i^{-1}\mu_i(A_i^*)\} = \sup\left\{\min_{i\in I}\{\alpha_i^{-1}\mu_i(A_i)\}\colon P = \{A_i\}_{i=1}^n \in \mathscr{P}\right\}.$$

Thus the definition of the α-optimal partition is correct.

A method of obtaining the α-optimal partition was given by Legut and Wilczyński [**5**].

COROLLARY 2. *There exists a partition $P^0 = \{A_i^0\}_{i=1}^n \in \mathscr{P}$ such that*

$$M := \sum_{i=1}^n \mu_i(A_i^0) = \sup\left\{\sum_{i=1}^n \mu_i(A_i)\colon P = \{A_i\}_{i=1}^n \in \mathscr{P}\right\}.$$

Denote $p_i := \mu_i(A_i^0)$, $p := (p_1, p_2, \ldots, p_n)$ and

$$m := \min\{p_i[p_i - \alpha_i(M-1)]^{-1}\colon [p_i - \alpha_i(M-1)] > 0,\ i \in I\}.$$

The number M can be interpreted as the "cooperative" value of the fair division problem (cf. [**4**]). It is clear that if $\mu_i \neq \mu_j$ for some $i \neq j$ then $M > 1$.

Now we can state our main result.

THEOREM 3. $m \leq v^* \leq M$.

PROOF. At first we show the inequality $v^* \leq M$. Suppose that $v^* > M$. From the definition of the number v^* we obtain $\alpha_i^{-1}\mu_i(A_i^*) > M$, for all $i \in I$. Hence we have $\sum_{i=1}^n \mu_i(A_i^*) > M$. This inequality contradicts the definition of the number M.

To prove that $m \leq v^*$ we put $e_i = (0, \ldots, 1, \ldots, 0) \in R^n$ (1 is placed on the ith coordinate, $i \in I$). Clearly, $e_i \in \bar{\mu}(\mathscr{P})$, for all $i \in I$. Let V denote the convex hull of the set $\{p, \{e_i\}_{i=1}^n\}$. From Theorem 2 we have $V \subset \bar{\mu}(\mathscr{P})$. It is now sufficient to find a real number $t^* := \max\{t \in R\colon t\alpha \in V\}$. Solving the following system of $n+1$ linear equalities

$$\beta_i + \beta_{n+1}p_i = \alpha_i t, \qquad i \in I,$$
$$\sum_{i=1}^n \beta_i = 1,$$

with respect to $\beta_i \geq 0$, $i = 1, 2, \ldots, n+1$, we obtain $t^* = m$. Hence we conclude that $m \leq v^*$ and the proof is complete.

REMARK. Let $\alpha_i = n^{-1}$ for all $i \in I$. In this case we get from Theorem 3

$$p^*[p^* - n^{-1}(M-1)]^{-1} \leq v^* \leq M$$

where $p^* := \max\{p_i\colon i \in I\}$. Since $p^* \leq 1$ we obtain

$$n[n - (M-1)]^{-1} \leq np^*[np^* - (M-1)]^{-1}.$$

Finally, we have the result of Elton et al. [**3**]

$$[n-(M-1)]^{-1} \leq n^{-1}v^* \leq n^{-1}M.$$

EXAMPLE. Let $\mathscr{X} = [0,1]$ and $\mathscr{B}$ be the σ-algebra of Borel subsets of $[0,1]$. Let λ be the Lebesgue measure on $[0,1]$. We consider the case $n = 2$ with $\alpha_1 = \alpha_2 = 1/2$. Define $\mu_1 = \lambda$ and $\mu_2(A) = 2\lambda(A \cap [0,1/4)) + (2/3)\lambda(A \cap (1/4,1])$ for $A \in \mathscr{B}$. It is easy to verify that $p_1 = \mu_1((1/4,1]) = 3/4$, $p_2 = \mu_2([0,1/4)) = 1/2$ and $M = p_1 + p_2 = 5/4$. The result of Elton et al. [**3**] gives the following inequalities for the number v^*:

$$8/7 \leq v^* \leq 5/4,$$

but using the estimation from Theorem 3 we obtain

$$8/7 < 6/5 \leq v^* \leq 5/4.$$

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INSTYTUT MATEMATYKI, TECHNICAL UNIVERSITY OF WROCŁAW, 50-370 WROCŁAW, POLAND