INEQUALITIES FOR $\alpha$-OPTIMAL PARTITIONING
OF A MEASURABLE SPACE

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ABSTRACT. An $\alpha$-optimal partition $\{A_i^*\}_{i=1}^n$ of a measurable space according to $n$ nonatomic probability measures $\{\mu_i\}_{i=1}^n$ is defined. A two-sided inequality for $v^* = \min\alpha_i^{-1}\mu_i(A_i^*)$ is given. This estimation generalizes and improves a result of Elton et al. [3].

1. Introduction. Let $(\mathcal{X}, \mathcal{B})$ denote a measurable space and let $\{\mu_i\}_{i=1}^n$ be nonatomic probability measures defined on the same $\sigma$-algebra $\mathcal{B}$. Let $\mathcal{P}$ stand for the set of all measurable partitions $P = \{A_i\}_{i=1}^n$ of $\mathcal{X}$, $A_i \cap A_j = \emptyset$, for all $i \neq j$.

DEFINITION. A partition $P^* = \{A_i^*\}_{i=1}^n \in \mathcal{P}$ is said to be an $\alpha$-optimal if

$$\min\{\alpha_i^{-1}\mu_i(A_i^*)\} = \sup\left\{\min\{\alpha_i^{-1}\mu_i(A_i)\} : P = \{A_i\}_{i=1}^n \in \mathcal{P}\right\},$$

where $I = \{1, 2, \ldots, n\}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ is a vector satisfying $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i > 0$ for all $i \in I$.

The main purpose of this paper is to give as good as possible estimation of the number $v^* = \min\{\alpha_i^{-1}\mu_i(A_i^*)\}$ by suitable inequalities. Using other simpler methods we generalize and improve the result of Elton et al. [3]. In §2 we state and prove the main theorem.

2. The main result. The problem of the $\alpha$-optimal partitioning of a measurable space $(\mathcal{X}, \mathcal{B})$ can be interpreted as the well-known problem of fair division (cf. [1, 6]) of an object $\mathcal{X}$ with unequal weights. Here, each $\mu_i, i \in I$, represents the individual evaluation of sets from $\mathcal{B}$. We also assume in this problem that $\{\mu_i\}_{i=1}^n$ are nonatomic probability measures. Dividing the object $\mathcal{X}$ fairly we are interested in giving the $i$th person a set $A_i \in \mathcal{B}$ such that $\mu_i(A_i) > \alpha_i$, for all $i \in I$.

Under assumptions given above, Dubins and Spanier [1] proved the following

THEOREM 1. Assume that $\mu_i \neq \mu_j$ for some $i \neq j$. Then there exists a partition $P = \{A_i\}_{i=1}^n \in \mathcal{P}$ such that $\mu_i(A_i) > \alpha_i$ for all $i \in I$.

The proof of Theorem 1 can be derived from the following result of Dvoretzky et al. [2] (cf. [1]).

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THEOREM 2. Let $\bar{\mu}: \mathcal{P} \to R^n$ denote the division vector valued function defined by
\[ \bar{\mu}(P) = (\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n)) \in R^n, \quad P = \{A_i\}_{i=1}^n \in \mathcal{P}. \]
Then the range $\bar{\mu}(\mathcal{P})$ of $\bar{\mu}$ is convex and compact in $R^n$.

From Theorem 2 we conclude

COROLLARY 1. There exists a partition $P^* = \{A_i^*\}_{i=1}^n \in \mathcal{P}$ such that
\[ v^* = \min_{i \in I} \{ \alpha_i^{-1} \mu_i(A_i^*) \} = \sup \left\{ \min_{i \in I} \{ \alpha_i^{-1} \mu_i(A_i) \} : P = \{A_i\}_{i=1}^n \in \mathcal{P} \right\}. \]
Thus the definition of the $\alpha$-optimal partition is correct.

A method of obtaining the $\alpha$-optimal partition was given by Legut and Wilczyński [5].

COROLLARY 2. There exists a partition $P^0 = \{A_i^0\}_{i=1}^n \in \mathcal{P}$ such that
\[ M := \sum_{i=1}^n \mu_i(A_i^0) = \sup \left\{ \sum_{i=1}^n \mu_i(A_i) : P = \{A_i\}_{i=1}^n \in \mathcal{P} \right\}. \]
Denote $p_i := \mu_i(A_i^0), p := (p_1, p_2, \ldots, p_n)$ and
\[ m := \min \{ p_i [\alpha_i(1 - \alpha_i(M - 1))]^{-1} : [p_i - \alpha_i(1 - \alpha_i(M - 1))] > 0, \ i \in I \}. \]
The number $M$ can be interpreted as the "cooperative" value of the fair division problem (cf. [4]). It is clear that if $\mu_i \neq \mu_j$ for some $i \neq j$ then $M > 1$.

Now we can state our main result.

THEOREM 3. $m \leq v^* \leq M$.

PROOF. At first we show the inequality $v^* \leq M$. Suppose that $v^* > M$. From the definition of the number $v^*$ we obtain $\alpha_i^{-1} \mu_i(A_i^*) > M$, for all $i \in I$. Hence we have $\sum_{i=1}^n \mu_i(A_i^*) > M$. This inequality contradicts the definition of the number $M$.

To prove that $m \leq v^*$ we put $e_i = (0, \ldots, 1, \ldots, 0) \in R^n$ (1 is placed on the $i$th coordinate, $i \in I$). Clearly, $e_i \in \bar{\mu}(\mathcal{P})$, for all $i \in I$. Let $V$ denote the convex hull of the set $\{p, \{e_i\}_{i=1}^n\}$. From Theorem 2 we have $V \subset \bar{\mu}(\mathcal{P})$. It is now sufficient to find a real number $t^* := \max \{ t \in R : t \alpha \in V \}$. Solving the following system of $n + 1$ linear equalities
\[ \beta_i + \beta_{n+1} p_i = \alpha_i t, \quad i \in I, \]
\[ \sum_{i=1}^n \beta_i = 1, \]
with respect to $\beta_i \geq 0$, $i = 1, 2, \ldots, n + 1$, we obtain $t^* = m$. Hence we conclude that $m \leq v^*$ and the proof is complete.

REMARK. Let $\alpha_i = n^{-1}$ for all $i \in I$. In this case we get from Theorem 3
\[ p^*[p^* - n^{-1}(M - 1)]^{-1} \leq v^* \leq M \]
where $p^* := \max \{p_i : i \in I\}$. Since $p^* \leq 1$ we obtain
\[ n[n - (M - 1)]^{-1} \leq np^*[np^* - (M - 1)]^{-1}. \]
Finally, we have the result of Elton et al. [3]

\[\left[n - (M - 1)\right]^{-1} \leq n^{-1}v^* \leq n^{-1}M.\]

EXAMPLE. Let \(\mathcal{B} = [0,1]\) and \(\mathcal{B}\) be the \(\sigma\)-algebra of Borel subsets of \([0,1]\). Let \(\lambda\) be the Lebesgue measure on \([0,1]\). We consider the case \(n = 2\) with \(\alpha_1 = \alpha_2 = 1/2\). Define \(\mu_1 = \lambda\) and \(\mu_2(A) = 2\lambda(A \cap [0,1/4]) + (2/3)\lambda(A \cap (1/4,1])\) for \(A \in \mathcal{B}\). It is easy to verify that \(p_1 = \mu_1((1/4,1]) = 3/4\), \(p_2 = \mu_2([0,1/4)) = 1/2\) and \(M = p_1 + p_2 = 5/4\). The result of Elton et al. [3] gives the following inequalities for the number \(v^*\):

\[8/7 < v^* < 5/4,\]

but using the estimation from Theorem 3 we obtain

\[8/7 < 6/5 < v^* < 5/4.\]

REFERENCES


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