A NEW CHARACTERIZATION OF TREES

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ABSTRACT. It is proved that a continuum is a tree if and only if for each pair of nondegenerate subcontinua \( K \) and \( L \) with \( K \subset L \), it follows that \( K \) contains a cutpoint of \( L \).

A connected Hausdorff space \( X \) is said to be tree-like [1] if each pair of distinct points can be separated by a third point, i.e., if \( x, y \in X \) and \( x \neq y \) then there exists \( z \in X \) so that \( X \setminus \{z\} = A \cup B \) where \( A \) and \( B \) are disjoint open sets, \( x \in A \) and \( y \in B \). A tree is a compact tree-like space. It is well known [5] that a continuum is a dendrite if and only if it is a metrizable tree.

It is a theorem of R. L. Moore [3] that a metrizable continuum \( X \) is a dendrite if and only if each nondegenerate subcontinuum of \( X \) contains uncountably many cutpoints of \( X \). In the setting of Hausdorff continua this theorem does not generalize to yield a characterization of trees. (For a counterexample, see [4, Example 9].) Moore’s theorem is now sixty years old, and trees have been intensively studied for over forty years and are quite well understood. Therefore, it is surprising that a suitable variation of Moore’s theorem, permitting a characterization of trees, has so far eluded discovery. In this note we obtain such a variation.

A point \( z \) of a connected space \( X \) is a cutpoint of \( X \) provided \( X \setminus \{z\} \) is not connected. A continuum is a compact connected Hausdorff space. A continuum \( X \) is hereditarily unicoherent if, for each pair of subcontinua \( A \) and \( B \) it follows that \( A \cap B \) is connected. An ordered continuum is a nondegenerate continuum with exactly two non-cutpoints. Finally, a continuum \( X \) is said to satisfy property \( T \) provided, for each pair of nondegenerate subcontinua \( K \) and \( L \) with \( K \subset L \), it follows that \( K \) contains a cutpoint of \( L \).

THEOREM 1. A continuum is a tree if and only if it satisfies the property \( T \).

PROOF. Suppose \( X \) is a tree and that \( K \) and \( L \) are nondegenerate subcontinua with \( K \subset L \). Let \( x \) and \( y \) be elements of \( K \). Since \( L \) is a tree there exists \( z \in L \) such that \( L \setminus \{z\} = L_1 \cup L_2 \), a separation, with \( x \in L_1 \) and \( y \in L_2 \). But \( K \) is a subcontinuum of \( L \), so \( z \in K \). Hence \( K \) contains a cutpoint of \( L \).

Now assume \( X \) is a continuum satisfying property \( T \). The proof that \( X \) is a tree will be divided into 4 parts.

1. \( X \) is hereditarily unicoherent. For suppose there are subcontinua \( A \) and \( B \) of \( X \) such that \( A \cap B = P \cup Q \) where \( P \) and \( Q \) are nonempty separated sets. Then...
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\(B - A\) contains a continuum \(I\) which is irreducible between \(P\) and \(Q\). Let \(x \in I - A\) and suppose \(C\) and \(D\) are components of \(I - \{x\}\) which meet \(P\) and \(Q\), respectively. (It is possible that \(C = D\).) Then \(\overline{C \cup D} = I\) by the irreducibility of \(I\), and \(C \cup D \cup A\) is a connected set which is dense in \(A \cup I - \{x\}\). Accordingly, no point of \(I - A\) is a cutpoint of \(A \cup I\). Therefore, if \(I_1\) is a nondegenerate subcontinuum of \(I\) which is disjoint from \(A\), then \(I_1\) contains no cutpoint of \(A \cup I\), contradicting the hypothesis that \(X\) satisfies property \(T\). Therefore \(X\) is hereditarily unicoherent.

2. It follows at once that if \(\emptyset \neq S \subset X\) then \(X\) contains a unique subcontinuum irreducible about \(S\). If \(S = \{x, y\}\) we denote that subcontinuum by \([x, y]\). Note that if \(t \in [x, y]\) then \([x, y] = [x, t] \cup [t, y]\) by irreducibility. Moreover, if \([x, y]\) is contained in a subcontinuum \(L\), then it is clear that \([x, y] - \{x, y\}\) contains a nondegenerate subcontinuum. Therefore, by property \(T\), the set \([x, y] - \{x, y\}\) contains a cutpoint of \(L\). We will use this fact repeatedly in the rest of the proof.

3. If \(x\) and \(y\) are distinct points of \(X\) then \([x, y]\) is an ordered continuum. It will suffice to show that if \(z \in [x, y] - \{x, y\}\) then \(z\) is a cutpoint of \([x, y]\). If the contrary obtains then \([x, z] = [x, y] \cup [z, y]\) where \([x, z] \cap [z, y]\) is nondegenerate. Since \(X\) is hereditarily unicoherent, \([x, z] \cap [z, y]\) is a continuum and therefore contains a cutpoint \(w\) of \([x, y]\), by property \(T\). Therefore \([x, y] = [x, w] \cup [w, y]\) with \([x, w]\) \cap \([w, y]\) = \(\{w\}\), and we may assume \(z \in [x, w] - \{w\}\). Therefore \([x, z] \subset [x, w]\), and since \(w \in [x, z]\) it follows that \([x, z] = [x, w]\); however \([w, y]\) is properly contained in \([z, y]\) and so \(z \notin [w, y]\). Again by property \(T\), the set \([z, w] - \{z, w\}\) contains a cutpoint \(w'\) of \([x, y]\). Since \(w' \in [w, z] \subset [z, w]\), it follows that \(w \in [w', y]\). Thus we have

\([x, y] = [x, w'] \cup [w', w] \cup [w, y]\)

and so \(z \in [z, w'] \cup [w', w] = [x, w]\). Because \([x, z] = [x, w]\) it is clear that \(z \notin [x, w']\) and therefore \(z \in [w', w]\). Applying property \(T\) again, the set \([w', w] - \{w', w\}\) contains a cutpoint \(w''\) of \([x, y]\). Clearly \([w', w] = [w', w''] \cup [w'', w]\) and \([w', w''] \cap [w'', w] \subset [x, w''] \cap [w'', w] = \{w''\}\). Since \(z \in [w', w]\), either \(z \in [w', w'']\) or \(z \notin [w'', w]\). But if \(z \in [w', w'']\) then \([x, z] \subset [w', w'']\), which is impossible. And if \(z \in [w'', w]\) then \([z, w] \subset [w'', w]\) and hence \(w \notin [z, w]\), which is also impossible. We have arrived at a contradiction and therefore \(z\) is a cutpoint of \([x, y]\).

4. \(X\) is a tree. Suppose \(x\) and \(y\) are distinct elements of \(X\) and \(z \in [x, y] - \{x, y\}\).

We will show that \(z\) is a cutpoint of \(X\) and that \(z\) separates \(x\) and \(y\) in \(X\).

For each \(t \in X - \{z\}\) let \(A_t\) denote the union of all ordered continua which contain \(t\) and are contained in \(X - \{z\}\). (That is, \(A_t\) is the Hausdorff space analog of an arc-component of \(X - \{z\}\).) Note that \([z, t] - \{z\} \subset A_t\). Also, if \(s \in A_t\) then \(A_s = A_t\), and hence we may regard \(t\) as an arbitrary point of \(A_t\). Suppose there exists a net \(t_\alpha\) with values in \(X - A_t\) and \(\lim t_\alpha = t\). Let \(Z = \bigcup \{[t_\alpha, t]\}\); clearly \(Z\) is a continuum. Since \(t_\alpha \notin A_t\) it follows that \(z \in [t_\alpha, t]\) for each \(\alpha\) and hence \(z \in Z\). Therefore \([z, t] \subset Z\) so that by property \(T\), \([z, t] - \{z, t\}\) contains a cutpoint \(p\) of \(Z\). Thus \(Z\) is the union of proper subcontinua \(Z_1\) and \(Z_2\) with \(Z_1 \cap Z_2 = \{p\}\). Moreover, because \([p, t] \subset [z, t] - \{z\}\), it is clear that \(p \in A_t\). If \(z \in Z_1\) and \(t \in Z_2\) then eventually \(t_\alpha \in Z_2\) so that \([t_\alpha, p] \subset Z_2\). It follows that \(t_\alpha \in A_t\), a contradiction. Therefore we may assume that both \(z\) and \(t\) lie in \(Z_1\). If there exists \(\alpha_1\) such that \(t_{\alpha_1} \in Z_2\) then \(p \in [t_{\alpha_1}, z]\). Since \(p \in A_t\) it follows that \(t_{\alpha_1} \in A_t\), a contradiction. Consequently \(t_\alpha \in Z_1\) for all \(\alpha\) and therefore \(Z \subset Z_1\). But this
contradicts the assumption that $Z_1$ is a proper subcontinuum of $Z$. Therefore, no such net $t_\alpha$ can exist and we conclude that each set $A_t$ is open. Since $z$ cuts $[x, y]$ it is clear that $A_x$ and $A_y$ are disjoint. That is to say, $X - \{z\}$ is not connected and $z$ separates $x$ and $y$ in $X$.

I have learned recently that E. Tymchatyn and W. Bula have obtained another proof of Theorem 1.

Let us say that a continuum satisfies property $M$ if each of its nondegenerate subcontinua contains uncountably many cutpoints of $X$. One sees easily that a continuum satisfies property $T$ if and only if each subcontinuum satisfies property $M$. The natural question arises: suppose $P$ is a property of metrizable continua which is equivalent to being a dendrite; among Hausdorff continua when is the condition that each subcontinuum satisfies $P$ equivalent to being a tree? We conclude this note with two more properties of this sort.

Let us say that a continuum satisfies property $W$ if each of its points is either a cutpoint or an endpoint. Wilder [6] proved that a metrizable continuum is a dendrite if and only if it satisfies property $W$, but this result does not generalize to Hausdorff continua.

**THEOREM 2.** A continuum is a tree if and only if each of its nondegenerate subcontinua satisfies property $W$.

**PROOF.** Every subcontinuum of a tree is a tree and it is known [4] that each tree satisfies property $W$. For the converse it is sufficient, in view of Theorem 1, to show that if $X$ is a continuum each of whose nondegenerate subcontinua satisfies property $W$, and if $K$ and $L$ are nondegenerate subcontinua of $X$ with $K \subset L$, then $K$ contains a cutpoint of $L$. Now $K$ must have a cutpoint $p$ and so $p$ is not an endpoint of $K$. It follows that $p$ is not an endpoint of $L$ so that, by property $W$, $p$ is a cutpoint of $L$.

If $X$ is a continuum, $Y$ is a subcontinuum and $p \in Y$, let $\theta(p, Y)$ denote the order of $p$ in $Y$, i.e., $\theta(p, Y)$ is the least cardinal number $n$ such that if $U$ is a neighborhood of $p$ then there exists a neighborhood $V$ of $p$ with $V \subset U$ and so that $|\partial V \cap Y| \leq n$. The component number of $p$ in $Y$, denoted $\alpha(p, Y)$, is the number of components of $Y - \{p\}$. It is a theorem of Menger [2] that a metrizable continuum $X$ is a dendrite if and only if $\alpha(p, X) = 0$ for each $p \in X$ for which either of these numbers is finite. Again, this result fails for Hausdorff continua.

**THEOREM 3.** If $X$ is a continuum then $X$ is a tree if and only if for each subcontinuum $Y$ of $X$ it follows that $\theta(p, Y) = \alpha(p, Y)$ for each $p \in Y$ for which either of these numbers is finite.

**PROOF.** It is known [4] that each tree satisfies the condition. For the converse it suffices to show that the condition implies property $W$. If $p$ is a noncutpoint of $Y$ then $\alpha(p, Y) = 1$; hence $\theta(p, Y) = 1$, i.e., $p$ is an endpoint of $Y$.

**REFERENCES**


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