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A NEW CHARACTERIZATION OF TREES

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ABSTRACT. It is proved that a continuum is a tree if and only if for each pair of nondegenerate subcontinua K and L with $K \subset L$, it follows that K contains a cutpoint of L.

A connected Hausdorff space X is said to be *tree-like* [**1**] if each pair of distinct points can be separated by a third point, i.e., if $x, y \in X$ and $x \neq y$ then there exists $z \in X$ so that $X - \{z\} = A \cup B$ where A and B are disjoint open sets, $x \in A$ and $y \in B$. A *tree* is a compact tree-like space. It is well known [**5**] that a continuum is a dendrite if and only if it is a metrizable tree.

It is a theorem of R. L. Moore [**3**] that a metrizable continuum X is a dendrite if and only if each nondegenerate subcontinuum of X contains uncountably many cutpoints of X. In the setting of Hausdorff continua this theorem does not generalize to yield a characterization of trees. (For a counterexample, see [**4**, Example 9].) Moore's theorem is now sixty years old, and trees have been intensively studied for over forty years and are quite well understood. Therefore, it is surprising that a suitable variation of Moore's theorem, permitting a characterization of trees, has so far eluded discovery. In this note we obtain such a variation.

A point z of a connected space X is a *cutpoint* of X provided $X - \{z\}$ is not connected. A *continuum* is a compact connected Hausdorff space. A continuum X is *hereditarily unicoherent* if, for each pair of subcontinua A and B it follows that $A \cap B$ is connected. An *ordered continuum* is a nondegenerate continuum with exactly two non-cutpoints. Finally, a continuum X is said to satisfy *property T* provided, for each pair of nondegenerate subcontinua K and L with $K \subset L$, it follows that K contains a cutpoint of L.

THEOREM 1. *A continuum is a tree if and only if it satisfies the property T.*

PROOF. Suppose X is a tree and that K and L are nondegenerate subcontinua with $K \subset L$. Let x and y be elements of K. Since L is a tree there exists $z \in L$ such that $L - \{z\} = L_1 \cup L_2$, a separation, with $x \in L_1$ and $y \in L_2$. But K is a subcontinuum of L, so $z \in K$. Hence K contains a cutpoint of L.

Now assume X is a continuum satisfying property T. The proof that X is a tree will be dividied into 4 parts.

1. *X is hereditarily unicoherent.* For suppose there are subcontinua A and B of X such that $A \cap B = P \cup Q$ where P and Q are nonempty separated sets. Then

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$\overline{B-A}$ contains a continuum I which is irreducible between P and Q. Let $x \in I - A$ and suppose C and D are components of $I - \{x\}$ which meet P and Q, respectively. (It is possible that $C = D$.) Then $\overline{C} \cup \overline{D} = I$ by the irreducibility of I, and $C \cup D \cup A$ is a connected set which is dense in $A \cup I - \{x\}$. Accordingly, no point of $I - A$ is a cutpoint of $A \cup I$. Therefore, if I_1 is a nondegenerate subcontinuum of I which is disjoint from A, then I_1 contains no cutpoint of $A \cup I$, contradicting the hypothesis that X satisfies property T. Therefore X is hereditarily unicoherent.

2. It follows at once that *if* $\varnothing \neq S \subset X$ *then* X *contains a unique subcontinuum irreducible about* S. If $S = \{x, y\}$ we denote that subcontinuum by $[x, y]$. Note that if $t \in [x, y]$ then $[x, y] = [x, t] \cup [t, y]$ by irreducibility. Moreover, if $[x, y]$ is contained in a subcontinuum L, then it is clear that $[x, y] - \{x, y\}$ contains a nondegenerate subcontinuum. Therefore, by property T, the set $[x, y] - \{x, y\}$ contains a cutpoint of L. We will use this fact repeatedly in the rest of the proof.

3. *If* x *and* y *are distinct points of* X *then* $[x, y]$ *is an ordered continuum.* It will suffice to show that if $z \in [x, y] - \{x, y\}$ then z is a cutpoint of $[x, y]$. If the contrary obtains then $[x, y] = [x, z] \cup [z, y]$ where $[x, z] \cap [z, y]$ is nondegenerate. Since X is hereditarily unicoherent, $[x, z] \cap [z, y]$ is a continuum and therefore contains a cutpoint w of $[x, y]$, by property T. Therefore $[x, y] = [x, w] \cup [w, y]$ with $[x, w] \cap [w, y] = \{w\}$, and we may assume $z \in [x, w] - \{w\}$. Therefore $[x, z] \subset [x, w]$, and since $w \in [x, z]$ it follows that $[x, z] = [x, w]$; however $[w, y]$ is properly contained in $[z, y]$ and so $z \notin [w, y]$. Again by property T, the set $[z, w] - \{z, w\}$ contains a cutpoint w' of $[x, y]$. Since $w' \in [z, w] \subset [x, w]$, it follows that $w \in [w', y]$. Thus we have

$$[x, y] = [x, w'] \cup [w', w] \cup [w, y]$$

and so $z \in [x, w'] \cup [w', w] = [x, w]$. Because $[x, z] = [x, w]$ it is clear that $z \notin [x, w']$ and therefore $z \in [w', w]$. Applying property T again, the set $[w', w] - \{w', w\}$ contains a cutpoint w'' of $[x, y]$. Clearly $[w', w] = [w', w''] \cup [w'', w]$ and $[w', w''] \cap [w'', w] \subset [x, w''] \cap [w'', y] = \{w''\}$. Since $z \in [w', w]$, either $z \in [w', w'']$ or $z \in [w'', w]$. But if $z \in [w', w'']$ then $[x, z] \subset [x, w''] \neq [x, w]$, which is impossible. And if $z \in [w'', w]$ then $[z, w] \subset [w'', w]$ and hence $w' \notin [z, w]$, which is also impossible. We have arrived at a contradiction and therefore z is a cutpoint of $[x, y]$.

4. X *is a tree.* Suppose x and y are distinct elements of X and $z \in [x, y] - \{x, y\}$. We will show that z is a cutpoint of X and that z separates x and y in X.

For each $t \in X - \{z\}$ let A_t denote the union of all ordered continua which contain t and are contained in $X - \{z\}$. (That is, A_t is the Hausdorff space analog of an arc-component of $X - \{z\}$.) Note that $[z, t] - \{z\} \subset A_t$. Also, if $s \in A_t$ then $A_s = A_t$, and hence we may regard t as an arbitrary point of A_t. Suppose there exists a net t_α with values in $X - A_t$ and $\lim t_\alpha = t$. Let $Z = \overline{\bigcup\{[t_\alpha, t]\}}$; clearly Z is a continuum. Since $t_\alpha \notin A_t$ it follows that $z \in [t_\alpha, t]$ for each α and hence $z \in Z$. Therefore $[z, t] \subset Z$ so that by property T, $[z, t] - \{z, t\}$ contains a cutpoint p of Z. Thus Z is the union of proper subcontinua Z_1 and Z_2 with $Z_1 \cap Z_2 = \{p\}$. Moreover, because $[p, t] \subset [z, t] - \{z\}$, it is clear that $p \in A_t$. If $z \in Z_1$ and $t \in Z_2$ then eventually $t_\alpha \in Z_2$ so that $[t_\alpha, p] \subset Z_2$. It follows that $t_\alpha \in A_t$, a contradiction. Therefore we may assume that both z and t lie in Z_1. If there exists α_1 such that $t_{\alpha_1} \in Z_2$ then $p \in [t_{\alpha_1}, z]$. Since $p \in A_t$ it follows that $t_{\alpha_1} \in A_t$, a contradiction. Consequently $t_\alpha \in Z_1$ for all α and therefore $Z \subset Z_1$. But this contradicts the assumption that Z_1 is a proper subcontinuum of Z. Therefore, no such net t_α can exist and we conclude that each set A_t is open. Since z cuts $[x,y]$ it is clear that A_x and A_y are disjoint. That is to say, $X - \{z\}$ is not connected and z separates x and y in X.

I have learned recently that E. Tymchatyn and W. Bula have obtained another proof of Theorem 1.

Let us say that a continuum satisfies *property M* if each of its nondegenerate subcontinua contains uncountably many cutpoints of X. One sees easily that a continuum satisfies property T if and only if each subcontinuum satisfies property M. The natural question arises: suppose P is a property of metrizable continua which is equivalent to being a dendrite; among Hausdorff continua when is the condition that each subcontinuum satisfies P equivalent to being a tree? We conclude this note with two more propertiees of this sort.

Let us say that a continuum satisfies *property W* if each of its points is either a cutpoint or an endpoint. Wilder [**6**] proved that a metrizable continuum is a dendrite if and only if it satisfies property W, but this result does not generalize to Hausdorff continua.

THEOREM 2. *A continuum is a tree if and only if each of its nondegenerate subcontinua satisfies property W.*

PROOF. Every subcontinuum of a tree is a tree and it is known [**4**] that each tree satisfies property W. For the converse it is sufficient, in view of Theorem 1, to show that if X is a continuum each of whose nondegenerate subcontinua satisfies property W, and if K and L are nondegenerate subcontinua of X with $K \subset L$, then K contains a cutpoint of L. Now K must have a cutpoint p and so p is not an endpoint of K. It follows that p is not an endpoint of L so that, by property W, p is a cutpoint of L.

If X is a continuum, Y is a subcontinuum and $p \in Y$, let $\theta(p,Y)$ denote the *order of* p *in* Y, i.e., $\theta(p,Y)$ is the least cardinal number n such that if U is a neighborhood of p then there exists a neighborhood V of p with $\overline{V} \subset U$ and so that $|\partial V \cap Y| \leq n$. The *component number* of p *in* Y, denoted $\alpha(p,Y)$, is the number of components of $Y - \{p\}$. It is a theorem of Menger [**2**] that a metrizable continuum X is a dendrite if and only if $\alpha(p,X) = \theta(p,X)$ for each $p \in X$ for which either of these numbers is finite. Again, this result fails for Hausdorff continua.

THEOREM 3. *If X is a continuum then X is a tree if and only if for each subcontinuum Y of X it follows that $\theta(p,Y) = \alpha(p,Y)$ for each $p \in Y$ for which either of these numbers is finite.*

PROOF. It is known [**4**] that each tree satisfies the condition. For the converse it suffices to show that the condition implies property W. If p is a noncutpoint of Y then $\alpha(p,Y) = 1$; hence $\theta(p,Y) = 1$, i.e., p is an endpoint of Y.

REFERENCES

1. H. Kok, *Connected orderable spaces*, Math. Centre Tracts, no. 49, Amsterdam, 1973.
2. K. Menger, *Kurventheorie*, 2nd ed., Chelsea, 1967.
3. R. L. Moore, *Concerning the cutpoints of continuous curves and other closed and connected point-sets*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 101–106.

4. L. E. Ward, Jr., *Recent developments in dendritic spaces and related topics*, Studies in Topology, Academic Press, San Francisco, 1975, pp. 601–647.
5. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942.
6. R. L. Wilder, *Concerning continuous curves*, Fund. Math. **7** (1925), 340–377.

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