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AN AREA THEOREM FOR A ONE-DIMENSIONAL SEMIDIRECT EXTENSION OF HOMOGENEOUS GROUPS

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ABSTRACT. Let N be a homogeneous group [**3**] and let $\{\delta_a : a \in A = R^+\}$ be the group of dilations of N. We prove an area theorem for harmonic functions w.r.t. a class of second-order left-invariant hypoelliptic differential operators L on the semidirect product $S = NA$ with $axa^{-1} = \delta_a(x)$, $a \in A$, $x \in N$.

Introduction. Area theorems have been proved for the harmonic functions on the half-space [**8**] and for the harmonic functions w.r.t. the Laplace-Beltrami operator on symmetric spaces: rank-one case in [**5**], the product of two rank-one spaces in [**6**].

In [**5**] A. Korányi and R. Putz have proved that, for a harmonic function f on a symmetric space G/K, $G = \overline{N}AK$ and a measurable set $M \subset \overline{N}$ the following are equivalent:

(i) f is a.e. nontangentially bounded on M;

(ii) the area integral is a.e. finite on M;

(iii) f is a.e. nontangentially covergent on M.

In the present paper we prove the implication (i) $\to$ (ii) in a more general situation.

We start with an arbitrary homogeneous group N with dilations $\{\delta_a : a \in A = R^+\}$ and we form the semidirect product $S = NA$, $xax'a' = x\delta_a(x')aa'$, $x, x' \in N$, $a, a' \in A$. We equip S with a left-invariant Riemannian metric in such a way that if N comes from the Iwasawa decomposition, then S becomes a rank-one symmetric space. As in [**2**] we consider not only the Laplace-Beltramni operator but a large class of second-order left-invariant hypoelliptic operators. In the case of a symmetric space G/K this covers second-order hypoelliptic operators which commute with the $\overline{N}A$-action and anihilate constants. Since all the left-invariant Riemannian metrics on S give proportional volume elements, the final result depends only on the group structure not on the Riemannian metric.

The method used in the present paper is a modification of the one of A. Korányi and R. Putz [**5**]. We have replaced the mean value theorem for symmetric spaces by Harnack inequality [**1**] and the Green theorem by the divergence theorem.

Whether the remaining implications hold in the case considered here is an open question. The proof by A. Korányi and R. Putz in [**5**] is based on very specific properties of symmetric spaces and the Laplace-Beltrami operator and we do not see any method of replacing it by another argument.

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Area theorem. A simply connected nilpotent Lie group N is called homogeneous [**3**] if there is a basis $E_1', \dots, E_n'$ of the Lie algebra $\underline{n}$ of N and numbers $1 = d_1 \le \cdots \le d_n$ such that for $a > 0$ the map $E_j' \to a^{d_j} E_j'$ extends to an automorphism δ_a of $\underline{n}$. Identifying X and $\exp X$ we see that δ_a is an automorphism of N. It is called a *dilation.*

On N there is a function

$$N \ni x \to |x| \in R^+$$

(homogeneous norm), which is C^∞ outside $x = e$, $\delta_a(x) = a|x|$, $|x| = 0$ iff $x = e$, $|xy| \le \gamma(|x| + |y|)$ for some $\gamma \ge 1$. If meas denotes the Lebesgue measure on N and $B(r) = \{x \in N \colon |x| < r\}$, we have $\operatorname{meas}(B(r)) = c_1 r^Q$ for a universal constant c_1 and $Q = d_1 + \cdots + d_n$.

In this paper we study the solvable group $S = NA$ which is the semidirect product of N and the group of dilations $A = R^+$ with $axa^{-1} = \delta_a(x)$, $a \in A$, $x \in N$. Let $E_1, \dots, E_n$ be left-invariant vector fields on S corresponding to $E_1', \dots, E_n'$ and E_0 a vector field defined by

$$E_0 f(s) = \frac{d}{dt} f(se^t)|_{t=0}, \qquad e^t \in A, \ s \in S.$$

Then E_j applied to the function $S \ni s \to a(s)^Q \subset R^+$ is equal to

$$\begin{matrix} Qa(s)^Q & \text{if } j = 0 \text{ and to} \\ 0 & \text{otherwise.} \end{matrix} \tag{1}$$

We define a Riemannian metric $\langle \ , \ \rangle$ on S assuming that $E_0, \dots, E_n$ are orthonormal. The volume element η is then proportional to the left-invariant measure on S, that is to $a^{-Q-1} dx da$.

If $N = R$, $n = 1$, $d_1 = 1$ we obtain the hyperbolic plane—the simplest example of a symmetric space. In fact the hyperbolic plane may be identified with the group

$$S = R \times R^+ = \{(x, a) \colon x \in R, \ a \in R^+\},$$

where

$$(x, a)(x_1, a_1) = (x + ax_1, aa_1)$$

and the left-invariant metric is $a^{-2}\delta_{ij}$. In this case $E_0 = \partial/\partial a$, $E_1 = a\partial/\partial x$ and $\eta = a^{-2} dx da$.

Applying the formula

$$\langle \nabla_{Y_1} Y_2, Y_3 \rangle = \tfrac{1}{2}(\langle [Y_1, Y_2], Y_3 \rangle - \langle [Y_2, Y_3], Y_1 \rangle + \langle [Y_3, Y_1], Y_2 \rangle)$$

for the Riemannian connection ∇ (cf. eg. [**7**]) we have

$$\nabla_{E_i} E_i = d_i E_0.$$

Since $(\operatorname{div} Y)_s = \operatorname{tr}(w \to \nabla_w Y)$, $w \in T_s S$,

$$\operatorname{div} Y = \sum_{i=0}^n E_i \varphi_i - Q\varphi_0, \tag{2}$$

where $Y = \sum_{i=0}^n \varphi_i E_i$, $\varphi_i \in C^\infty(S)$. As in [**5**] we define the regions

$$\Gamma_\alpha^\tau(x) = \{ya \colon |y^{-1}x| < \alpha a, \ a < \tau\}$$

and we write

$$W_\alpha^\tau(M) = \bigcup_{x\in M} \Gamma_\alpha^\tau(x)$$

for a measurable set $M \subset N$. We say that a function f is *nontangentially bounded* at x if there are $\alpha, \tau > 0$ such that f is bounded in $\Gamma_\alpha^\tau(x)$.

Let L be a second-order left-invariant operator L on S of the form

$$L = (E_0 + X_0)^2 + X_1^2 + \cdots + X_m^2 + X,$$

where $X_0, X_1, \ldots, X_m \in \underline{n}$ and $E_0 + X_0, X_1, \ldots, X_m$, X generate the Lie algebra of S. We say that a function f is harmonic iff $Lf = 0$. We write

$$D(f) = ((E_0 + X_0)f)^2 + (X_1 f)^2 + \cdots + (X_m f)^2.$$

The aim of this paper is the following

THEOREM. *If $f \in C^\infty(S)$ is harmonic and nontangentially bounded at almost every point of a measurable set $M \subset N$, then for every $\alpha, \tau > 0$*

$$\int_{\Gamma_\alpha^\tau(x)} D(f)\, d\eta < \infty$$

for a.e. $x \in M$.

We may assume that M is bounded. The proof of the Theorem is based on four lemmas. The first is a simple modification of Lemma 5 of [**4**], the second has been proved in [**5**].

LEMMA 1. *Let $\alpha, \beta, \sigma, \varepsilon$ be positive numbers and M a measurable set. There are $\tau > 0$ and a measurable set $D \subset M$,* $\operatorname{meas}(M \backslash D) < \varepsilon$ *such that*

$$\Gamma_\alpha^\tau(y) \subset \bigcup_{x\in M} \Gamma_\beta^\sigma(x)$$

for $y \in D$.

LEMMA 2. *Let M be a measurable set. If $\int_{W_\alpha^\tau(M)} D(f) a^Q\, d\eta < \infty$ then $\int_{\Gamma_\alpha^\tau(x)} D(f)\, d\eta < \infty$ for a.e. $x \in M$.*

LEMMA 3. *Let f be a bounded harmonic function on $\Gamma_\beta^\sigma(x)$. Then for every $\alpha < \beta$, $\tau < \sigma$, $D(f)$ and $E_i f$, $i = 0, \ldots, n$ are bounded in $\Gamma_\alpha^\tau(x)$.*

PROOF. Since L is left-invariant, it is sufficient to consider only the case $x = e$. Let $\Gamma_\beta^\sigma(e) = \Gamma_\beta^\sigma$, $|g|_\infty = \sup_{x\in\Gamma_\beta^\sigma} |g(x)|$ and $K = \{xa\colon |x| < \alpha a,\ \frac{1}{2}\tau < a < \tau\}$. Then $bK = \{xa\colon |x| < \alpha a,\ \frac{1}{2} b\tau < a < \tau b\}$ and $\Gamma_\alpha^\tau = \bigcup_{b<1} bK$. The Harnack inequality [**1**] implies that there is a constant c_2 such that for every harmonic function g on Γ_β^σ, $\sup_K D(g)$ and $\sup_K |E_i g|$, $i = 0, \ldots, n$, are bounded by $c_2 |g|_\infty$. To estimate $D(f)$, $|E_i f|$ on bK, $b < 1$, we take the harmonic function $g(s) = f(bs)$ defined on $b^{-1}\Gamma_\beta^\sigma \supset \Gamma_\beta^\sigma$. Then

$$\sup_{bK} D(f) = \sup_K D(g), \quad \sup_{bK} |E_i f| = \sup_K |E_i g|, \qquad i = 0, \ldots, n.$$

Hence $\sup_{bK} D(f)$, $\sup_{bK} |E_i f|$, $i = 0, \ldots, n$ are bounded by $c_2 |f|_\infty$.

We write L in the form

$$L = \sum_{i,j=0}^{n} \alpha^{ij} E_i E_j + \sum_{i=0}^{n} \alpha^i E_i, \ \alpha^{ij} = \alpha^{ji},$$

and define

$$L^* = \sum_{i,j=0}^{n} \alpha^{ij} E_i E_j + \sum_{i=0}^{n} (-\alpha^i - 2Q\alpha^{0i}) E_i + Q(Q\alpha^{00} + \alpha^0).$$

Then

$$\int_S (Lf) g \, d\eta = \int_S f(L^* g) \, d\eta, \qquad f, g \in C_c^\infty(S),$$

because

$$\int_S g E_i f \, d\eta = -\int_S f E_i g \, d\eta \quad \text{for } i > 0 \text{ and}$$
$$\int_S g E_0 f \, d\eta = -\int_S f E_0 g \, d\eta + Q \int_S f g \, d\eta.$$

Taking into account (2) we can easily check that

$$\text{(3)} \qquad fLg - gL^* f = \operatorname{div} \left(\sum_{i=0}^{n} \left(\sum_{j=0}^{n} \alpha^{ij} (f E_j g - g E_j f) + (Q\alpha^{0i} + \alpha^i) f g \right) E_i \right).$$

The next step is to replace the regions $W_\alpha^\tau(M)$ by regions W_l, on which we can apply Stoke's Theorem [**9**, p. 100]. We take the same regions W_l as A. Korányi and R. B. Putz in [**5**]. Let $\{x_j\}_{j=1}^\infty$ be a countable dense sequence in M and

$$W_l = \bigcup_{1 \le j \le l} (\Gamma_\alpha^\tau(x_j) \cap \{xa \colon a > 1/l\}),$$
$$B_{l,0} = \partial W_l \cap \{xa \colon a = \tau\},$$
$$B_{l,1} = \partial W_l \cap \{xa \colon 1/l \le a < \tau\}.$$

Then $W_l \subset W_{l+1}$, $\bigcup_{l=1}^\infty W_l = W_\alpha^\tau(M)$ and we have the following lemma (Lemma 5 of [**5**]).

LEMMA 4. *Let $d\rho$ be the surface element on W_l, d/dn the outward pointing unit normal to $B_{l,1}$ and dn the left-invariant volume element on N. Then there exist constants $c_3, c_4 > 0$ independent of l such that*

$$a^Q \le -c_3 \, da^Q/dn, \quad -(da^Q/dn) \, d\rho = c_4 \pi^*(dn),$$

where $\pi \colon B_{l,1} \to N$ by $\pi(xa) = x$.

THE PROOF OF THE THEOREM. Let $\varepsilon > 0$ and $M_m = \{x \in M \colon |f|, |E_i f|, i = 0, \dots, n, D(f) < m \text{ on } \Gamma_{1/m}^1(x)\}$. Since $M_m \subset M_{m+1}$ and by Lemma 3 $\operatorname{meas}(M) = \operatorname{meas}(\bigcup_{m=1}^\infty M_m)$, we can choose M_m such that $\operatorname{meas}(M \backslash M_m) < \varepsilon/2$. Let $\alpha > 0$, $\tau > 0$. By Lemma 1 there is a measurable set D such that $\operatorname{meas}(M_m \backslash D) < \varepsilon/2$ and

$$\text{(4)} \qquad |f|, D(f), |E_i f|, \le m \text{ on } \bigcup_{x \in D} \Gamma_\alpha^\tau(x), \quad i = 0, \dots, n.$$

Now in view of Lemma 2 it is sufficient to show that $\int_{W_\alpha^\tau(D)} D(f)a^Q\,d\eta < \infty$. Since by (1) $L^*a^Q = 0$, we have $2a^Q D(f) = a^Q Lf^2 - (L^*a^Q)f^2$ for a harmonic function f. On the other hand (3) combined with (1) gives

$$a^Q Lf^2 - (L^*a^Q)f^2 = \operatorname{div}\left(\sum_{i=0}^n \left(\sum_{j=0}^n \alpha^{ij}E_j f^2 + \alpha^i f^2\right) a^Q E_i\right).$$

Applying the divergence theorem

$$\int_{W_t} (\operatorname{div} Y)\,d\eta = \int_{\partial W_t} \left\langle Y, \frac{d}{dn}\right\rangle d\rho$$

we obtain

$$\int_{W_t} A^Q D(f)\,d\eta = \frac{1}{2}\int_{\partial W_t} \left\langle \sum_{i=0}^n \left(\sum_{j=0}^n \alpha^{ij}E_j f^2 + \alpha^i f^2\right) a^Q E_i, \frac{d}{dn}\right\rangle d\rho.$$

Hence by (4)

$$\int_{W_t} a^Q D(f)\,d\eta < \frac{1}{2}c_5 \int_{\partial W_t} a^Q\,d\rho$$

for a constant c_5 and by Lemma 4 the right side of (5) is uniformly bounded.

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