ON THE GLAUBERMAN CORRESPONDENCE

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Abstract. In this paper we give an elementary proof of the \( p \)-group case of Glauberman's correspondence.

1. INTRODUCTION

Let \( p \) be a prime number.

Let \( S \) be a \( p \)-group acting on a finite \( p' \)-group \( G \). Let \( C = C_G(S) \) and \( \text{Irr}_S(G) = \{ \chi \in \text{Irr}(G) \mid \chi^s = \chi, \ \forall s \in S \} \).

It is well known that the map \( K \to K \cap C \) defines a bijection from the set of \( S \)-invariant conjugacy classes of \( G \) onto the set of conjugacy classes of \( C \) [2, 13.10].

Let \( R \) be the full ring of algebraic integers in \( C \), let \( M \) be a maximal ideal of \( R \) containing \( pR \), and set \( F = R/M \). Let \( \ast : R \to F \) the canonical homomorphism.

For \( \chi \in \text{Irr}(G) \), the map defined on \( Z(F[G]) \) by \( \lambda_{\chi}(\widehat{K}) = \omega_{\chi}(\widehat{K})^\ast \) is an algebra homomorphism from \( Z(F[G]) \) to \( F \). Note that \( \lambda_{\chi}(\widehat{K}) = (\chi(x)|K/\chi(1))^\ast \) for \( x \in K \).

2. TWO LEMMAS

2.1. Lemma. Let \( \chi \in \text{Irr}_S(G) \). We define \( \delta_{\chi} : Z(F[C]) \to F \) by setting \( \delta_{\chi}(\widehat{K} \cap C) = (\chi(x)|K/\chi(1))^\ast \) for \( K \) \( S \)-invariant conjugacy class of \( G \) and \( x \in K \cap C \).

Then \( \delta_{\chi} \) is an algebra homomorphism.

Proof. Since \( \delta_{\chi}(\widehat{K} \cap C) = \lambda_{\chi}(\widehat{K}) \), it suffices to show that

\[
\delta_{\chi}(\widehat{K_i \cap C K_j \cap C}) = \lambda_{\chi}(\widehat{K_i \cap K_j})
\]

for \( K_i, K_j, S \)-invariant conjugacy classes of \( G \).

Write \( \widehat{K}_1, \ldots, \widehat{K}_h \) for the \( S \)-invariant conjugacy classes, and \( K_{h+1,1}, \ldots, K_{h,1}, \ldots, K_{h+1,a}, \ldots, K_{h,a_1} \) for the rest, where \( K_{h+j,1}, \ldots, K_{h+j,a_i} \) is an \( S \)-orbit. Note that \( a_j = p^{b_j} > 1 \).

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It is clear that we can write
\[
\hat{K}_i \hat{K}_j = \sum_{k=1}^{h} a_{ijk} K_k + \sum_{l=1}^{l} b_{ijl}(\sum_{m=1}^{m} \hat{K}_{h+l,m}).
\]

Fix \(x_k \in K_i \cap C\). Since \(a_{ijk} = |\{(x, y) \in K_i \times K_j : xy = x_k\}|\) and \(\{(x, y) \in K_i \times K_j : xy = x_k\} \equiv |\{(x, y) \in K_i \cap C \times K_j \cap C : xy = x_k\}| \mod p\), we have that \(K_i \cap CK_j \cap C = \sum_{k=1}^{h} a_{ijk} K_k \). Now, since \(\lambda_x\) is constant over each \(S\)-orbit, we have
\[
\lambda_x(\hat{K}_i \hat{K}_j) = \sum_{k=1}^{h} a_{ijk} \lambda_x(K_k) + \sum_{l=1}^{l} b_{ijl} \lambda_x(\hat{K}_{h+l,l})
\]
\[
= \sum_{k=1}^{h} a_{ijk} \lambda_x(K_k)
\]
\[
= \sum_{k=1}^{h} a_{ijk} \delta_x(K_k \cap C)
\]
\[
= \delta_x(\sum_{k=1}^{h} a_{ijk} K_k \cap C) = \delta_x(K_i \cap CK_j \cap C).
\]

The following result is well known. We give the proof to make it clear that no results on \(p\)-blocks are needed for this paper.

2.2. Lemma. Suppose \(p\) does not divide \(|G|\). The maps \(\lambda_x\) for \(x \in \text{Irr}(G)\) are distinct and are all the algebra homomorphisms from \(Z(F[G])\) to \(F\).

Proof. \(F[G]\) is a direct sum of full matrix rings over \(F\), so \(Z(F[G]) \cong F^k\) where \(k = \text{cl}(G)\). There are thus \(k\) algebra homomorphisms \(Z(F[G]) \to F\) and it is enough to show that the \(\lambda_x\) are distinct.

Let \(e_x = \chi(1) \sum_{g \in G} \chi(x^{-1})^* g \in Z(F[G])\). Then \(\lambda_x(e_x) = |G|\neq 0\) and \(\lambda_y(e_x) = 0\) for \(x \neq y\). The result now follows.

3. The Glauberman correspondence

Notation. Given \(x \in \text{Irr}_S(G)\), since \(\delta_x\) is an algebra homomorphism \(Z(F[C]) \to F\), it follows that there exists a unique \(\widetilde{\chi} \in \text{Irr}(C)\) such that \(\delta_x = \lambda_{\widetilde{\chi}}\). Thus for \(x \in C\), we have
\[
(\chi(x)|K|/\chi(1))^* = (\widetilde{\chi}(x)|K \cap C|/\widetilde{\chi}(1))^*\text{, where }K = \text{Cl}_G(x).
\]

Since \(|K| \equiv |K \cap C| \neq 0 \mod p\), this gives \(\widetilde{\chi}(1)\chi(x) \equiv \chi(1)\widetilde{\chi}(x) \mod M\), for all \(x \in C\).

3.1. Theorem. The map \(\text{Irr}_S(G) \to \text{Irr}(C)\) defined by \(\chi \to \widetilde{\chi}\) is a bijection. Also, \([\chi_C , \widetilde{\chi}] \equiv \pm 1 \mod p\,\text{, and }[\chi_C , \theta] \equiv 0 \mod p\) for \(\widetilde{\chi} \neq \theta \in \text{Irr}(C)\).

Proof. Let \(\chi \in \text{Irr}_S(G)\) and \(\theta \in \text{Irr}(C)\). Then
\[
|C|\widetilde{\chi}(1)[\chi_C , \theta] = \widetilde{\chi}(1) \sum_{x \in C} \chi(x)\theta(x^{-1}) \equiv \chi(1) \sum_{x \in C} \widetilde{\chi}(x)\theta(x^{-1})
\]
\[
= \chi(1)[\widetilde{\chi} , \theta]|C| \mod M.
\]
Since $p$ does not divide $|C|$, this gives $\tilde{\chi}(1)[\chi_C, \theta] \equiv \chi(1)[\tilde{\chi}, \theta] \mod p$. Since $p$ does not divide $\tilde{\chi}(1)$, taking $\theta \neq \tilde{\chi}$, this gives $[\chi_C, \theta] \equiv 0 \mod p$, and taking $\theta = \tilde{\chi}$ gives

$$\tilde{\chi}(1)[\chi_C, \tilde{\chi}] \equiv \chi(1) \not\equiv 0 \mod p.$$  

Thus $\tilde{\chi}$ is the unique irreducible constituent of $\chi_C$ with multiplicity not divisible by $p$.

Now let $\chi \cdot \varphi \in \text{Irr}_S(G)$. Then

$$|G|[\chi, \varphi] = \sum_{x \in G} \chi(x)\varphi(x^{-1}) \equiv \sum_{x \in C} \chi(x)\varphi(x^{-1}) = |C|[\chi_C, \varphi_C] \mod p$$

(using that $\chi, \varphi$ are $S$-invariant). Also, $|G| \equiv |C| \not\equiv 0 \mod p$, and so $[\chi, \varphi] \equiv [\chi_C, \varphi_C] \mod p$.

Since $\chi_C = [\chi_C, \tilde{\chi}]\tilde{\chi} + p\Delta$ and $\varphi_C = [\varphi_C, \tilde{\varphi}]\tilde{\varphi} + p\Phi$, we have that $[\chi, \varphi] = [\chi_C, \tilde{\chi}][\varphi_C, \tilde{\varphi}]\tilde{\chi} \mod p$. (2)

This shows that our map is injective.

If $\alpha \in \text{Irr}(C)$ is not in the image of our map, then $[\chi_C, \alpha] = [\chi, \alpha^G] \equiv 0 \mod p$ for all $\chi \in \text{Irr}_S(G)$. Since $\alpha^G$ is $S$-invariant, $[\alpha^G, \varphi^s] = [\alpha^G, \varphi]$ for all $\varphi \in \text{Irr}(G)$, for all $s \in S$. This implies that $p$ divides $\alpha^G(1)$, a contradiction.

Finally, taking $\chi = \varphi$ in (2), we get $1 \equiv [\chi_C, \tilde{\chi}]^2 \mod p$ and so $[\chi_C, \tilde{\chi}] \equiv \pm 1 \mod p$ as claimed.

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**References**