ON THE PERIODIC POINTS OF A TYPICAL CONTINUOUS FUNCTION

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Abstract. Let $n$ and $k$ be arbitrary natural numbers. We prove that for a typical continuous function $f$, every neighborhood of any periodic point of $f$ with period $n$ contains periodic points of $f$ with period $n \cdot k$.

The purpose of this paper is to examine the structure of the set of periodic points of typical continuous functions. We shall denote by $C$ the set of continuous functions mapping the interval $[0,1]$ into itself. This set becomes a metric space with the supremum metric. By the term "typical continuous function" we mean that the set of all functions having the property under consideration is a residual subset of the metric space $C$.

In [1] S. J. Agronsky, A. M. Bruckner, and M. Laczkovich proved that for a typical continuous function $f$ any neighborhood of a periodic point contains periodic points of arbitrarily large periods. In this paper we prove that this result is true if "arbitrarily large periods" is replaced by "period $k \cdot n$ ($k = 1, 2, \ldots$)" (Theorem 1). As a consequence, for a typical continuous $f$ we have that $P^n_f$, the set of periodic points with period $n$, is uncountable, dense in itself, but not closed. Also, $P^n_f$ is a residual subset of $\text{Fix}(f^n)$.

In [1] it is proved that, for a typical continuous $f$, $\text{Fix}(f^n)$ is nowhere dense and perfect. In this paper we prove (Theorem 2) that $\text{Fix}(f^n)$ is bilaterally strongly $\Phi$-porous. For $f \in C$ and $n \in \mathbb{N}$ we define $f^n(x)$ by $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$. $P^n_f$ denotes the set of periodic points of $f$ with period $n$. We denote $\text{Fix}(f^n) = \{x : f^n(x) = x\}$; then $P^n_f = \text{Fix}(f^n) \setminus \bigcup_{k \neq n} \text{Fix}(f^k)$. In particular $P^n_f$ is a relatively open subset of $\text{Fix}(f^n)$. For $f \in C$ and $\epsilon > 0$ denote

$$B(f, \epsilon) = \{g \in C : \|f - g\| < \epsilon\}.$$

Theorem 1. Let $n, k \in \mathbb{N}$ be arbitrary. Then for a typical continuous function $f$, every neighborhood of any periodic point of $f$ with period $n$ contains periodic points of $f$ with period $n \cdot k$.

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Proof. Fix $n \in \mathbb{N}$. Let $A_0 = \{ f \in C : \text{Fix}(f^n) \text{ nowhere dense} \}$ and $A_i = \{ f \in C : \text{every neighborhood of any periodic point of } f \text{ with period } i \text{ contains, for every } m \in \mathbb{N}, \text{ periodic points of } f \text{ with period } m \cdot i \}$ $(i = 1, 2, \ldots)$.

We prove by induction that each $A_i$ is residual. In [1] it is proved that the set $A_0$ is a residual subset of $C$. Assume that $\bigcap_{0 \leq i < n} A_i$ is a residual subset of $C$. We shall prove that $A_n$ is also residual. If $f \in \bigcap_{0 \leq i < n} A_i$ and $I$ is an arbitrary open interval then $P^n_f \cap I = \emptyset$ implies $\text{Fix}(f^n) \cap I = \emptyset$. For $k \in \mathbb{N}$ and $\epsilon > 0$ we define the set of $Q_k^\epsilon$ by

$$Q_k^\epsilon = \{ f \in C : \text{for any } x_0 \in P^n_f \text{ there is } x \in P^{k\cdot n}_f \text{ such that } |x - x_0| < \epsilon \}. $$

It will be sufficient to show that each $Q_k^\epsilon$ contains a set which is open and dense. Let $f \in \bigcap_{0 \leq i < n} A_i$ and let $\epsilon' > 0$ be arbitrary. We prove that there exist $g \in B(f, \epsilon')$ and $\eta > 0$ such that $B(g, \eta) \subset Q_k^\epsilon$. Let $\{ a_i \}_{i=0}^l \subset [0, 1] \setminus \text{Fix}(f^n)$ be a finite sequence such that $a_0 = 0$, $a_l = 1$ and $0 < a_{i+1} - a_i < \epsilon$ for every $0 \leq i < l$. Put $J_i = [a_{i-1}, a_i]$ $(i = 1, \ldots, l)$. Choose a finite sequence $\{ b_j \}_{j=1}^z \subset P^n_f$ as follows: if $J_i \cap P^n_f \neq \emptyset$ $(i = 1, \ldots, l)$ then there should exist an $x_0 \in J_i \cap P^n_f$ such that the sequence $\{ a_i \}_{i=0}^l$ contains the orbit of $x_0$. Let $z' = z/n$. We can choose $b_1, b_2, \ldots, b_z$ so that they are contained in different orbits. Choose $\eta_1 \in \mathbb{R}$ so that

(i) $\eta_1 > 0$,

(ii) $\eta_1 < \frac{1}{2} \cdot \min_{0 < i < j < l} \{ |a_i - a_j|, |b_j - b_j|, |a_i - b_j| \}$,

(iii) $\eta_1 < \epsilon'/3$, and

(iv) if for some $i$ $J_i \cap P^n_f = \emptyset$ and $\|g - f\| < 3\eta_1$ then $J_i \cap P^n_g = \emptyset$.

This is possible since $f \in \bigcap_{i<n} A_i$. Let $0 < u < \eta_1$ be such that if $|x - y| < u$ then $|f(x) - f(y)| < \eta_1$. Then $[b_i - u/2, b_i + u/2] \cap [b_j - u/2, b_j + u/2] = \emptyset$ for every $1 \leq i < j \leq z$.

We can now define the function $g \in C$. First we define $g$ on

$$\bigcup_{i=1}^{z'} [f^i(b_m) - u/2, f^i(b_m) + u/2] \quad (m = 1, \ldots, z')$$

using the following procedure. Let $1 \leq m \leq z'$ be fixed but arbitrary and take $x_0 = b_m$. Put $I_j = [f^{j-1}(x_0) - u/2, f^{j-1}(x_0) + u/2]$. Let us define $g$ on the set $\bigcup_{j=1}^{z'} I_j$ so that the graph of $g$ on this set will be included in the squares $E_j = I_j \times I_{j+1}$ (see Figure 1). If $j > 1$ then let the graph of $g$ in $E_j$ be the increasing diagonal of $E_j$. (See Figure 1.) Let $x_1, \ldots, x_k$, $x_{k+1} = x_1$ be distinct points in $(0, u)$ and let the point $W_i$ be defined by $W_i = (x_i, x_{i+1})$ $(i = 1, \ldots, k)$; then $W_i \in L = [0, u] \times [0, u]$. Let $v = \min_{i \neq j} \frac{1}{2}|x_i - x_j|$ and define

$$B_i = \left[ x_i - \frac{v}{2}, x_i + \frac{v}{2} \right] \times \left[ x_{i+1} - \frac{v}{2}, x_{i+1} + \frac{v}{2} \right].$$
Let \( h : [0, u] \to [0, u] \) be piecewise linear and continuous such that the graph of \( h \) in each \( B_i \) is the diagonal of \( B_i \), if \( 1 \leq i < k \) then \( h \) is increasing in \( B_i \) but \( h \) is decreasing in \( B_k \). (See Figure 2.)

It is easy to see that if \( \eta = \nu / 10 \) and \( q \in B(h, \eta) \) then \( q \) has periodic points of period \( k \) in the interval \((x_i - u/2, x_i + u/2)\). Now we define the graph of \( g \) on \( I_1 \) by translating the graph of \( h \) into \( E_1 \) (using the translation that maps \( L \) onto \( E_1 \)). Thus we have defined \( g \) on \( \bigcup_{i=1}^{l} [b_i - u/2, b_i + u/2] \). Put \( g(a_i) = f(a_i) \) (\( i = 0, 1, \ldots, l \)). If \( J_i \cap P^n_f = \emptyset \) then let \( g|_{J_i} = f|_{J_i} \). For those \( x \)'s for which \( g \) has not yet been defined, let us define \( g \) so that \( \|f - g\| < \eta_1 \); by the definition of \( u \) this is possible. Now if \( P^n_f \cap J_i = \emptyset \) then \( P^n_g \cap J_i = \emptyset \) or else \( P^{n+k}_g \cap J_i \neq \emptyset \). Thus \( g \in B(f, \epsilon') \cap Q^e_k \) and it is easy to see that if \( \eta = \nu / 10 \) then \( B(g, \eta) \subset B(f, \epsilon') \cap Q^e_k \). It follows that \( Q^e_k \) contains
an open dense set, and since $\bigcap_{k} Q_k^e \subset A$ we have that $A$ is a residual subset of $C$. This completes the proof of the theorem.

We show that, for a typical continuous $f$, $P^k_f$ is a dense relatively open subset of $\text{Fix}(f^k)$ from which it will follow that, $P^k_f$ is a residual subset of $\text{Fix}(f^k)$ and $P^k_f$ is uncountable.

**Corollary 1.** For a typical continuous $f$, $P^k_f$ is a dense relatively open subset of $\text{Fix}(f^k)$.

*Proof.* If $x \in \text{Fix}(f^k)$ then there is $q_k$ such that $x \in P^q_f$. It follows from Theorem 1 that any neighborhood of $x$ contains periodic points with period $q \cdot k/q$. As we have seen in the introduction, $P^k_f$ is relatively open in $\text{Fix}(f^k)$.

**Corollary 2.** For a typical continuous $f$, $P^k_f$ is not closed for every $k > 1$.

*Proof.* It follows from Theorem 1 that for every $x \in \text{Fix}(f)$, $x \in \text{cl}(P^k_f)$ and $x \notin P^k_f$.

**Corollary 3.** For a typical continuous $f$, $P^k_f$ is dense in itself.

*Proof.* Since, for a typical $f$, $\text{Fix}(f^k)$ is dense in itself [1], the statement follows from Corollary 1.
Remark. For a typical $f$ there is $x \in \text{Fix}(f^n)$ such that in a suitable neighborhood of $x$, the periods of the periodic points are all multiples of $n$. Moreover, this is true for the points of a residual subset of $\text{Fix}(f^n)$. Indeed, let $f \in C$ and $I_0 \subset [0, 1]$ be an open interval and let $I_k = f^k(I_0)$. Define $P^n_k(f) = \{I_0: \text{ for every } 0 \leq 1 \neq j \leq k - 1, I_i \cap I_j = \emptyset \text{ and } I_n \subset I_0 \text{ and } \min\{|I_0|, \ldots, |I_n|\} < 1/k\}$. (See [1, Proposition 1].)

In [1] the authors proved that for a typical continuous function $f$, $\text{Fix}(f^n) \cap \bigcap_{k=1}^{\infty} G^n_k$ is a residual subset of $\text{Fix}(f^n)$. Put $B^n_f = \{x \in P^n_f: \text{ there is } V \text{ neighborhood of } x \text{ such that for any periodic point } y \in V \text{ there is } k \text{ such that that period of } y \text{ equal to } n \cdot k\}$. It is easy to see $P^n_f \cap \bigcap_{k=1}^{\infty} G^n_k \subset B^n_f$ and therefore $B^n_f$ is a residual subset of $\text{Fix}(f^n)$.

In [1] it is proved that for a typical continuous $f$ the set $\text{Fix}(f^k)$ is nowhere dense and perfect. In Theorem 2 we shall prove that for every porosity premeasure (i.e., a continuous map $\Phi: (0, 1] \rightarrow (0, 1]$), $\text{Fix}(f^k)$ is bilaterally strongly $\Phi$-porous for a typical continuous $f$. This means that for every $x \in \text{Fix}(f^k)$ there are sequences of intervals $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\infty}$ such that $I_n \subset (x - 1/n, x) \setminus \text{Fix}(f^k)$, $J_n \subset (x, x + 1/n) \setminus \text{Fix}(f^k)$ and

$$\lim_{n \to \infty} \frac{\text{dist}(x, I_n)}{\Phi(|I_n|)} = \lim_{n \to \infty} \frac{\text{dist}(x, J_n)}{\Phi(|J_n|)} = 0.$$ 

We shall apply the method used by P. Humke and M. Laczkovich in [2] to this problem.

The pair of sequences $(\alpha, \beta)$ is said to be proper if $\alpha = \{\alpha_n\}_{n=1}^{\infty}$, $\beta = \{\beta_n\}_{n=1}^{\infty}$ and $\beta_n \to 0$ and $0 < \alpha_n < \beta_n$. If $(\alpha, \beta)$ is a proper pair of sequences then the sequence $x = \{x_n\}_{n=1}^{\infty}$ is called an $(\alpha, \beta)$-sequence if $x_n \rightarrow x_0$ and $x_n - x_0 \leq \beta_n$ for each $n \in \mathbb{N}$. Let the natural number $l > 1$ and the proper pair $(\alpha, \beta)$ be fixed. (See [2, p. 245].) Put

$$B_N = \{f \in C: \text{ there is } \{x_i\}_{i=1}^{\infty}((\alpha, \beta)\text{-sequence such that } f^l(x_i) = x_i \text{ if } i \geq N\}.$$ 

Lemma. $B_N$ is nowhere dense.

Proof. First we shall prove that $B_N$ is closed. Let $\{f_k\}_{k=1}^{\infty} \subset B_N$ and $f_k \rightarrow f$ in $C$. Thus, for every fixed $k$ there is a sequence $x^k = \{x^k_n\}_{n=1}^{\infty}$ converging to $x^0_k$ such that $x^0_k$ is an $(\alpha, \beta)$-sequence and $f^0_k(x^0_k) = x^0_k$ if $i \geq N$. Using a diagonal procedure we obtain a subsequence $\{k_i\}_{i=1}^{\infty}$ such that for every $n$ there is $x^*_n$ such that $\{x^k_{n_i}\}_{i=1}^{\infty} \rightarrow x^*_n$ and $x^* = \{x^*_n\}_{n=1}^{\infty}$ is a $(\alpha, \beta)$-sequence. As $\{f_{k_i}\}_{i=1}^{\infty} \rightarrow f$ in $C$ and since for $n > N$ $f_{k_i}^l(x^k_{n_i}) = x^k_{n_i}$ it follows that $f^l(x^*_n) = x^*_n$ if $n > N$.

If $p$ is a polynomial then $\{x: p^l(x) = x\}$ is finite and hence $p \notin B_N$. Since the set of polynomials is dense in $C$, the complement of $B_N$ is dense and, as $B_N$ is closed, $B_N$ is nowhere dense.

Let $\Phi: (0, 1] \rightarrow (0, 1]$ be continuous arbitrary fixed.
Theorem 2. For a typical continuous \( f \), \( \text{Fix}(f^l) \) is bilaterally strongly \( \Phi \)-porous set.

Proof. Let \( (\alpha, \beta) \) be such that \( \alpha_n = \beta_{n+1}, \beta_n < 1/n \) and \( \Phi(\beta_n - \beta_{n+1}) \geq n\beta_{n+1} \). Let \( M = \{x: f^l(x) = x\} \). If \( f \in \bigcup_n B_n \) then for each \( x \in M \) there is a sequence \( \{n_i\} \to \infty \) such that \( M \cap [x + \alpha_n, x + \beta_n] = \emptyset \). Let \( J_i = [x + \alpha_n, x + \beta_n] \). Thus \( J_i \subset (x, x + 1/i) \backslash M \) and

\[
\text{dist}(x, J_i) = \alpha_{n_i} = \beta_{n_i+1} < \frac{\Phi(\beta_{n_i} - \beta_{n_i+1})}{n_i} = \frac{\Phi(|J_i|)}{n_i}
\]

and hence

\[
\lim_{n \to \infty} \frac{\text{dist}(x, J_i)}{\Phi(|J_i|)} = 0.
\]

Using a similar procedure we can find a sequence \( \{I_i\}_{i=1}^\infty \) such that

\[
I_i \subset (x - 1/i, x) \backslash M \quad \text{and} \quad \lim_{i \to \infty} \frac{\text{dist}(x, I_i)}{\Phi(|I_i|)} = 0.
\]

This completes the proof of the theorem.

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References


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