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ON MATRIX COEFFICIENTS OF THE REFLECTION REPRESENTATION

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ABSTRACT. We answer in the affirmative a question of Lusztig on the signs of certain matrix coefficients for the reflection representation of the Hecke algebra of a finite Weyl group. In the action of the Weyl group on the associated root system, the coefficients of the action of an element on a root are all of the same sign. It is shown that an appropriate generalization of this property holds for the reflection representation.

1.1 Let $(\mathscr{W}, S)$ be an indecomposable Coxeter pair where $\mathscr{W}$ is a finite Weyl group. Denote the Bruhat order on $\mathscr{W}$ by $<$ where e is minimal and let $l(\cdot)$ be the length function on $\mathscr{W}$. Let $\mathscr{H}$ be the Hecke algebra of $\mathscr{W}$ over the polynomial ring $\mathbf{Q}[u^{1/2}, u^{-1/2}]$ (cf. [10]). Following Kazhdan and Lusztig [10], we have two standard bases of $\mathscr{H}$, $\{T_w | w \in \mathscr{W}\}$ and $\{C_w | w \in \mathscr{W}\}$. Let Δ be a root system associated to $\mathscr{W}$ and for s in S, let α_s be the corresponding simple root. Then the reflection representation (π, E) of $\mathscr{H}$ has basis $\{d_t | t \in S\}$ and satisfies

$$\pi(T_s)d_t = \begin{cases} -d_s & \text{if } s = t, \\ ud_t - u^{1/2}\langle \alpha_t, \alpha_s \rangle d_s & \text{if } s \neq t, \end{cases} \qquad s, t \in S.$$

We refer the reader to [5] for the general properties of this representation. For x in $\mathscr{W}$ and s and t in S, let $\pi(x, s, t)$ be the t, s-matrix entry of $\pi(T_x)$, that is

$$\pi(T_x)d_s = \sum_{t \in S} \pi(x, s, t)\, d_t.$$

Theorem 1.1. *For x in $\mathscr{W}$ and s in S, the coefficients of the matrix entries $\pi(x, s, t)$, $t \in S$, are all of the same sign as $l(xs) - l(x)$.*

This theorem has been established previously in the case of A_n by Lusztig and in the other classical cases by Tiwari [12]. Our proof does not rely on a case by case argument and works for the exceptional systems as well as the classical.

1.2 Let $\sim_L$, $\sim_R$ and $\sim_{LR}$ be the left, right and two-sided equivalence relations on $\mathscr{W}$ defined in [10]. The equivalence classes of their relations are

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referred to, respectively, as left, right and two-sided cells. Let Γ_s be the left cell containing the simple reflection s and for s and t in S put $\Delta_{s,t} = \Gamma_s \cap (\Gamma_t)^{-1}$. For x in $\mathscr{W}$, define the matrix coefficients $h(x, s, t)$ by

$$\pi(C_x)d_s = \sum_{t \in S} h(x, s, t)\, d_t.$$

Lemma 1.2. *For s and t in S and w in $\Delta_{s,t}$ there is a positive integer $n_{w,s,t}$ so that $h(w, s, t) = -n_{w,s,t}(u^{-1/2} + u^{1/2})$.*

The proof of this lemma can be found in [6]. It is a straightforward computation based on the formula

$$\pi(C_s)d_t = \begin{cases} -(u^{1/2} + u^{-1/2})\, d_s & \text{if } s = t, \\ -\langle \alpha_t, \alpha_s \rangle\, d_s & \text{if } s \neq t, \end{cases} \qquad s, t \in S.$$

Let w_0 be the longest element of $\mathscr{W}$. For x and y in $\mathscr{W}$ let $P_{x,y}(u)$ be the Kazhdan-Lusztig polynomial as defined in [10]. As usual, the polynomials $Q_{x,y}$ are defined by $Q_{x,y} = P_{w_0 y, w_0 x}$. We will use the notations

$$u_x = u^{l(x)} \quad \text{and} \quad c(x, s) = \begin{cases} 1 & \text{if } x > xs, \\ 0 & \text{if } x < xs, \end{cases} \qquad x \in \mathscr{W},\ s \in S.$$

Also we will denote by $q \to \overline{q}$ the involution on $\mathbf{Q}[u^{1/2}, u^{-1/2}]$ given by $\overline{u^{1/2}} = u^{-1/2}$.

Theorem 1.2. *For x in $\mathscr{W}$ and s and t in S,*

$$\pi(x, s, t) = u_x \left[\overline{\sum_{w \in \Delta_{s,t}} n_{w,s,t} u_{ws}^{1/2} u^{c(x,s)} (Q_{w,xs}(u) - Q_{w,x}(u))} \right].$$

The proof of this theorem can also be found in [6]. It is based on a characterization of the two-sided cell $\mathbf{X}$ which corresponds to the reflection representation. This correspondence is described in [11]. In particular, it is shown in [6] that $\mathbf{X}$ is the two sided cell of $\mathscr{W}$ which contains all of the simple reflections and that $\mathbf{X}$ consists of exactly the nonidentity elements of $\mathscr{W}$ which have a unique reduced expression. We remark that we have not included several references needed for the proof of Theorem 1.2 and the pertinent facts about $\mathbf{X}$. A complete list of references can be found in [6].

From Theorem 1.2 it is clear that Theorem 1.1 will be proved as soon as we establish that the coefficients of $Q_{w,xs} - Q_{w,x}$ are nonnegative whenever $s \in S$, $w, x \in \mathscr{W}$ and $xs > x$.

2.1 Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ so that the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ is $\mathscr{W}$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$ with positive system Δ^+ compatible with S. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Borel subalgebra and let ρ be half the sum of the positive roots. For x in $\mathscr{W}$ we denote by M_x the $\mathfrak{g}$-Verma module with highest weight $-x\rho - \rho$. That is, M_x is the $\mathfrak{g}$-module induced from the one-dimensional $\mathfrak{b}$-module determined by the weight $-x\rho - \rho$. Each M_x has a unique irreducible quotient which we denote L_x.

For any simple module A and finite length module B, $[B : A]$ denotes the multiplicity of A in a composition series for B. Recall that the socle of a module M, $\mathrm{soc}^1(M)$, is the maximal semisimple submodule of M. Inductively, $\mathrm{soc}^i(M)$ is the pull back into M of the socle of $M/\mathrm{soc}^{i-1}(M)$. Put $\mathrm{soc}_i(M) = \mathrm{soc}^i(M)/\mathrm{soc}^{i-1}(M)$. The Loewy length of M, $ll(M)$, is the smallest i for which $\mathrm{soc}^i(M) = M$. From [4] (as well as other sources) we know that $ll(M_x) = l(x) + 1$.

Theorem 2.1 (Irving, [8]). *For w and x in $\mathscr{W}$,*

$$Q_{w,x}(u) = \sum_k [\mathrm{soc}_{1+l(w)+2k}(M_x) : L_w] u^k .$$

Irving's proof of Theorem 2.1 is based on the fact (also proved in [8]) that the socle filtration of M_x coincides with the radical filtration of M_x. We observe that a slightly different version of Theorem 2.1 was also obtained by Gabber and Joseph in [7]. In the Gabber-Joseph version the layers of the socle filtration of M_x are replaced by the layers of the Jantzen filtration of M_x (defined in [9]). This requires the use of the Jantzen filtration conjecture of which there is no published proof (a proof was announced by Beilinson in [1]). Irving's result 2.1 is a formal consequence of the Kazhdan-Lusztig conjectures as proved in [2 or 3]. Either way we now proceed to prove Theorem 1.1 using the hereditary property of the filtration. (In the case of the Jantzen filtration this is exactly the Jantzen conjecture.)

Suppose now that x and w are in $\mathscr{W}$, s is in S and $xs > x$. Then the coefficient of u^k in $Q_{w,xs}(u) - Q_{w,x}(u)$ is exactly

$$[\mathrm{soc}_{1+l(w)+2k}(M_{xs}) : L_w] - [\mathrm{soc}_{1+l(w)+2k}(M_x) : L_w].$$

However, M_x is a submodule of M_{xs} and the socle filtration is hereditary for submodules, i.e. $\mathrm{soc}^j(M_x) = \mathrm{soc}^j(M_{xs}) \cap M_x$ for all j. This shows that the coefficient of u^k is nonnegative. By Theorem 1.2 this completes the proof of Theorem 1.1. (Note: the above argument was observed in [8] as a corollary to Theorem 1.3. We include the argument for the sake of completeness.)

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