INFINITESIMAL PSEUDO-METRICS AND
THE SCHWARZ LEMMA

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Abstract. In this paper we investigate the relationship between infinitesimal pseudo-metrics introduced by N. Sibony and K. Azukawa. Also we prove a version of the Schwarz lemma for plurisubharmonic functions.

Introduction

Let $M$ be a complex manifold. Throughout the paper we shall be assuming that the dimension of $M$ is $n$. Sibony in [8] and Azukawa in [1] introduced infinitesimal pseudo-metrics on the tangent bundle $TM$ using families of bounded plurisubharmonic functions on $M$. We shall denote the pseudo-metrics by $S_M$ and $A_M$ respectively (the definitions of the pseudo-metrics are given in the next section). Both $S_M$ and $A_M$ contract holomorphic mappings and hence $C_M \leq S_M \leq K_M$ and $C_M \leq A_M \leq K_M$ where $C_M$ and $K_M$ denote the Carathéodory and Kobayashi infinitesimal pseudo-metrics respectively. The purpose of this paper is to study the relationship between $S_M$ and $A_M$.

In order to be able to state the main results we need to recall the definition of an extremal plurisubharmonic function, originally introduced in [6] and then studied in [2, 4].

Let $\mathcal{P}(M, p)$ denote the family of all negative plurisubharmonic functions $u$ on $M$ such that for any holomorphic chart $\varphi: U \to \varphi(U) \subset C^n$ where $p \in U \subset M$ and $\varphi(p) = 0$, the function $u \circ \varphi^{-1} - \log \| \cdot \|$ is bounded from above in a neighbourhood of 0. (By $\| \cdot \|$ we denote the Euclidean norm in $C^n$.)

Define

$$u_M^*(z, p) = \sup\{u(z) : u \in \mathcal{P}(M, p)\}.$$  \hfill (1)

It can be proved that $u_M^*(\cdot, p) \in \mathcal{P}(M, p)$. Moreover, if $M$ is an open subset of $C$ such that $u_M^*(z, p) \to 0$ as $z \to \partial M$, then $-u_M^*(\cdot, p)$ coincides with the Green function for $M$ with pole at $p$. For further properties of the extremal function $u_M^*$ and its applications in complex analysis see [6, 4, 2].
Let $C^2(p)$ denote the family of all functions which are of class $C^2$ in some neighbourhood of $p$.

We show the following.

**Theorem 1.** If $M$ is a complex manifold then $S_M \leq A_M$. If $M$ is a Stein manifold then $A_M$ is upper semicontinuous and hence $S_M^* \leq A_M$ (where the asterisk denotes the upper-semicontinuous regularization). If $\exp(2u_M(\cdot, p)) \in C^2(p)$ for some $p \in M$ then

\[(2) \quad S_M(\xi) = A_M(\xi) = \langle \mathcal{L}(\exp(2u_M(\cdot, p)))(p)\xi, \xi \rangle^{1/2}, \quad \xi \in T_pM,
\]

where $T_pM$ is the tangent space to $M$ at $p$ and $\langle \cdot, \cdot \rangle$ denotes the Levi form.

If $P_M$ is an infinitesimal pseudometric on $M$ and $p \in M$, the set $I_p(P_M) = \{v \in T_pM : P_M(v) < 1\}$ is called the indicatrix of $P_M$ at $p$. It can be shown that the indicatrices of $S_M$ are always convex (see [8]) and those of $A_M$ are—in general—only starlike circular. Therefore, it is easy to furnish examples of manifolds $M$ for which the two pseudo-metrics differ. For instance we can take a plurisubharmonic function $h : \mathbb{C}^n \to \mathbb{R}^+$ such that $h(\lambda z) = |\lambda|h(z)$ for $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$ and $M = \{z \in \mathbb{C}^n : h(z) < 1\}$ is not convex (e.g. $h(z_1, z_2) = \max\{|z_1|, |z_2|, 2\sqrt{|z_1z_2|}\}$ for $(z_1, z_2) \in \mathbb{C}^2$). Then $A_M(0, \xi) = h(\xi)$ (see [1]) and hence $A_M \neq S_M$ on $T_0M \equiv \mathbb{C}^n$. An example of a nonconvex $M$ for which $S_M = A_M$ is provided by the open annulus $M = \{z \in \mathbb{C} : r < |z| < R\}$ where $r > 0$, $R > 0$. The equality follows from Theorem 1 because $\exp(2u_M(\cdot, p)) \in C^2(M)$ (see [6]).

We also prove a version of the Schwarz lemma for plurisubharmonic functions. To state the result, we have to introduce some notation. Let $p \in M$ and let $\mathcal{S}(M, p)$ denote the family of all logarithmically plurisubharmonic functions $u : M \to [0, 1]$ such that $u(p) = 0$ and $u \in C^2(p)$. By $\mathcal{F}(M, p)$ we will denote the set of all plurisubharmonic functions $u$ on $M$ with the property that for each $z \in M \setminus \{p\}$ there exists a connected one-dimensional complex submanifold $N$ of $M$ such that $z, p \in N$ and the restriction of $u$ to $N \setminus \{p\}$ is harmonic.

**Theorem 2.** Let $\Omega$ be a relatively compact open connected subset of a Stein manifold $M$ and let $u_\Omega = u_\Omega(\cdot, p)$. If $v \in \mathcal{S}(\Omega, p)$ then

\[(3) \quad v \leq \exp(2u_\Omega) \quad \text{in} \quad \Omega \setminus \{p\}.
\]

Moreover if $\exp(2u_\Omega) \in C^2(p)$ then for all $\xi \in T_pM$

\[(4) \quad \langle \mathcal{L}v(p)\xi, \xi \rangle \leq \langle \mathcal{L}(\exp(2u_\Omega))(p)\xi, \xi \rangle.
\]

If $u_\Omega \in \mathcal{F}(\Omega, p)$ and the equality holds in (4) for all $\xi \in T_pM$ then $v \equiv \exp(2u_\Omega)$. If $n = 1$ and the equality holds in (3) for one $z \in \Omega$ then $v \equiv \exp(2u_\Omega)$.

When $\Omega$ is equal to the unit disc, $\exp(2u_\Omega(z)) = |z|^2$ and hence the theorem reduces to the version of the Schwarz lemma proved by Sibony in [8].
It is easy to notice that the last conclusion of the theorem is not true for \( n > 1 \). For instance, if \( \Omega \) is the open unit ball in \( \mathbb{C}^2 \), \( p \) is the origin and 
\[
v(z_1, z_2) = |z_1|^2 + a |z_2|^2 \tag{where \( a \in [0, 1) \) is fixed} \]
then \( v \neq \exp(2u_{\Omega}) = \| \cdot \|^2 \).

Note also that if \( n = 1 \), \( \exp 2u_{\Omega} \in \mathcal{S}^2(\{p\}) \). This is so, because \(-u_{\Omega}\) is the generalized Green function for \( \Omega \) with pole at \( p \).

Assume now that \( \Omega \subset M \) is such that \( u = u_{\Omega}(\cdot, p) \) is a \( \mathcal{G}^2 \)-function on \( \Omega \setminus \{p\} \) and \( u(z) \to 0 \) as \( z \) approaches the boundary of \( \Omega \). It is known that \( u \) satisfies the homogeneous Monge-Ampère equation \((dd^c u)^n = 0\) in \( \Omega \setminus \{p\} \) (see [6]). If we also assume that \( (dd^c u)^{-1} \neq 0 \) at each point of \( \Omega \setminus \{p\} \), then—in view of [3]—there is a foliation of \( \Omega \setminus \{p\} \) by one-dimensional complex manifolds with the property that the restriction of \( u \) to each of them is harmonic (an alternative proof can be found in [5]). As observed in [4], it follows from the maximum principle that if \( N \) is a leaf of the foliation, then \( p \in N \) and \( \overline{N} \cap \Omega \) is a one-dimensional analytic subvariety of \( \Omega \). It would be interesting to know under what condition \( p \) is a regular point of \( \overline{N} \cap \Omega \) for each leaf \( N \) of the foliation. (If this was the case, \( u \) would be a member of \( \mathcal{S}^2(\Omega, p) \).) Under the assumption that \( \Omega \) is a bounded strictly convex domain in \( \mathbb{C}^n \), an affirmative answer follows from Lempert’s study of the Kobayashi metric [7]. It has been conjectured by Demailly [4] that if \( \Omega \) is strictly pseudoconvex then the above conditions are also met—i.e. \( u \) is smooth on \( \Omega \setminus \{p\} \), \( dd^c u \) has constant rank \( n - 1 \) in \( \Omega \setminus \{p\} \) and the leaves of the associated foliation extend through \( p \) to complex submanifolds of \( \Omega \).

In the light of the above remarks, the assumption that \( u_{\Omega} \in \mathcal{S}(\Omega, p) \) does not seem to be too restrictive.

1. The Families of Functions \( \mathcal{P}(M, p) \) and \( \mathcal{S}(M, p) \)

In this section we shall establish the relationship between the two families of plurisubharmonic functions defined in the introduction.

The following lemma follows directly from the proof of a version of the Schwarz lemma obtained by Sibony in [8]. For the sake of completeness we give a proof here.

**Lemma 1.** Let \( D \subset \mathbb{C} \) be a neighbourhood of zero. If \( u \in \mathcal{S}(D, 0) \) then

\[ u(z) = \frac{1}{4}(\Delta u)(0)|z|^2 + o(|z|^2), \quad \text{as} \ z \to 0. \tag{5} \]

**Proof.** In view of Taylor’s formula for \( u \) at 0, \( v(z) = (u(z)/|z|^2)^* \) is subharmonic in \( D \) and

\[
\lim_{t \to 0^+} \frac{u(t \alpha, t \beta)}{t^2} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(0) \alpha^2 + \frac{\partial^2 u}{\partial x \partial y}(0) \alpha \beta + \frac{1}{2} \frac{\partial^2 u}{\partial y^2}(0) \beta^2
\]

for any \( \alpha + i \beta \) from the unit circle. Moreover, as closed line segments in \( \mathbb{C} \) are not thin, the limit on the left hand side of the above equality is \( v(0) \).

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taking $\alpha + i\beta$ equal to 1, $i$ and $(1 + i)/\sqrt{2}$ we conclude that
\[ \frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial y^2}(0) \quad \text{and} \quad \frac{\partial^2 u}{\partial x\partial y}(0) = 0. \]

Hence the Taylor expansion of $u$ at 0 looks exactly as stated in our lemma.

Let $M$ be a complex manifold and let $p \in M$. For $u \in \mathcal{C}^2(\{p\})$ one can define the Levi form $(\mathcal{L}u(p), \cdot)$ of $u$ at $p$ as follows. Let $\xi \in T_pM$ and let $\varphi: U \to \varphi(U) \subset \mathbb{C}^n$ be a chart on $M$ in a neighbourhood $U$ of $p$ such that $\varphi(p) = 0$. If $(\zeta_1, \ldots, \zeta_n) = d_p\varphi(\xi)$ then we put
\[ \langle \mathcal{L}u(p)\xi, \xi \rangle = \sum_{i,j=1}^n \frac{\partial^2 (u \circ \varphi^{-1})}{\partial z_i \partial \bar{z}_j}(0) \zeta_i \bar{\zeta}_j. \]

It is easy to show that this definition is independent of the choice of $\varphi$.

**Lemma 2.** If $u \in \mathcal{S}(M, p)$ then $\log \sqrt{u} \in \mathcal{P}(M, p)$. Moreover if $V \subset \mathbb{C}$ is a neighbourhood of the origin and $F: V \to M$ is a holomorphic mapping such that $F(0) = p$ and $F'(0) = \xi$ we have
\[ (6) \quad \langle \mathcal{L}u(p)\xi, \xi \rangle = \lim_{\lambda \to 0} \frac{(u \circ F)(\lambda)}{|\lambda|^2}. \]

**Proof.** Without loss of generality we may assume that $M$ is an open subset of $\mathbb{C}^n$ and $p = 0$. Since $\langle \mathcal{L}u(p)\xi, \xi \rangle = \frac{1}{2} \Delta(\lambda \to (u \circ F)(\lambda))|_{\lambda=0}$ (where $\lambda \in \mathbb{C}$), (5) implies that in a neighbourhood of the origin in $\mathbb{C}$
\[ (7) \quad (u \circ F)(\lambda) = \langle \mathcal{L}u(0)\xi, \xi \rangle |\lambda|^2 + o(|\lambda|^2). \]

(6) is implied directly by (7). Also from (7), applied to $F(\lambda) = \lambda \xi$, we deduce that Taylor’s expansion of $u$ about the origin in $\mathbb{C}^n$ has the following form:
\[ (8) \quad u(z) = \langle \mathcal{L}u(0)z, z \rangle + o(\|z\|^2). \]

The first conclusion of the lemma follows form (8).

It is interesting to notice that if $u \in \mathcal{PSH}(\mathbb{C}^n) \cap \mathcal{C}^2(\{0\})$ is such that
\[ (\dagger) \quad u(\lambda z) = |\lambda|^2 u(z) \]
for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$, then $u(z) = \langle \mathcal{L}u(0)z, z \rangle$ for all $z \in \mathbb{C}^n$ and hence $u^{1/2}$ is a seminorm. To see this, it is enough to apply the Laplace operator (with respect to $\lambda$) to both sides of $(\dagger)$. (See also the remarks following Theorem 1 in the Introduction.)

The families $\mathcal{S}(M, p)$ and $\mathcal{P}(M, p)$ have been used in the definitions of $S_M$ and $A_M$ respectively (see [8, 1, 2]). $S_M$ is defined by the formula
\[ (9) \quad S_M(\xi) = \sup \{\langle \mathcal{L}u(p)\xi, \xi \rangle^{1/2} : u \in \mathcal{S}(M, p)\}, \quad \xi \in T_pM. \]

For more information about the pseudo-metric $S_M$ see [8, 9].
Assume \( \xi \in T_p M \). Let \( V \subset C \) be a neighbourhood of the origin and let \( F: V \to M \) be a holomorphic mapping such that \( F(0) = p \) and \( F'(0) = \xi \). Then we put
\[
L_u[\xi] = \limsup_{\lambda \to 0, \lambda \neq 0} \exp(u \circ F)(\lambda)
\]
for any \( u \in \mathcal{P}(M, p) \). It can be proved that the above definition is independent of the choice of \( F \) (see [1, 2]). (If \( \exp(2u) \in \mathcal{C}^2(\{p\}) \), it follows directly from (6).)

Following Azukawa [1, 2] we define
\[
A_M(\xi) = \sup_{u \in \mathcal{P}(M, p)} \{ L_u[\xi] \}, \quad \xi \in T_p M.
\]

From the definition of the extremal function \( u_M = u_M(\cdot, p) \) we conclude (as in [2]) that for \( \xi \in T_p M \)
\[
A_M(\xi) = L_{u_M}[\xi].
\]

2. Semicontinuity of \( A_M \)

In this section we shall prove that if \( M \) is a Stein manifold then \( A_M \) is upper semicontinuous.

**Lemma 3.** If \( M \) is a Stein manifold then \( u_M: M \times M \to [-\infty, 0) \) is upper semicontinuous.

**Proof.** Since \( M \) is a Stein manifold, there exists a smooth plurisubharmonic function \( \psi: M \to R \) such that the set \( M_c = \{ z \in M: \psi(z) < c \} \) is relatively compact in \( M \) for each \( c \in R \). It follows from the definition of the extremal function \( u_M \) that the sequence \( \{ u_M \}_{j \in N} \) is decreasing and \( \lim_{j \to \infty} u_M = u_M \) (see also [2] and [4]). Furthermore, as each of the sets \( M_j \) is hyperconvex (in the sense of [4]), the functions \( u_{M_j}: M_j \times M_j \to [-\infty, 0) \) are continuous by Theorem 4.14 in [4]. This means that the limit function \( u_M \) is upper semicontinuous.

**Lemma 4.** If \( M \) is a complex manifold such that \( u_M: M \times M \to [-\infty, 0) \) is upper semicontinuous then \( A_M: TM \to [0, \infty) \) is also upper semicontinuous.

**Proof.** Take \( \xi_0 \in TM \). Let \( \pi: TM \to M \) be the canonical projection (i.e. \( \pi(x) = q \Leftrightarrow x \in T_q M \)). Let \( \phi: U \to \phi(U) \subset C^n \) be a holomorphic chart in a neighbourhood \( U \) of \( \pi(\xi_0) = p \). The chart \( \phi \) generates a chart \( \hat{\phi} \) on \( TM \) in the following way:
\[
\hat{\phi}: \pi^{-1}(U) \to \phi(U) \times C^n,
\]
\[
\hat{\phi}(\xi) = ((\phi \circ \pi)(\xi), a_1, \ldots, a_n) \Leftrightarrow \xi = \sum_{j=1}^{n} \frac{\partial}{\partial \phi_j} \pi(\xi).
\]
Define \( g : \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) by the formula \( g(\lambda, z, a) = \lambda a + z \) and set \( F(\lambda, \xi) = \varphi^{-1}(g(\lambda, \Phi(\xi))) \). Take a neighbourhood \( V_0 \) of \( \xi_0 \) and \( r_0 > 0 \) such that the mapping \( \lambda \rightarrow F(\lambda, \xi) \) is well defined if \( |\lambda| < r_0 \) and \( \xi \) is fixed in \( V_0 \). Let \( F_\xi \) denote this mapping. Clearly \( F_\xi(0) = \pi(\xi) \). In particular \( F_{\xi_0}(0) = p \).

Notice that if

\[
\xi = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \varphi_j} \bigg|_{\pi(\xi)},
\]

then \( (d_{\pi(\xi)} \varphi)(\xi) = (a_1, \ldots, a_n) \). Therefore

\[
d_0 F_\xi = (d_{\pi(\xi)} \varphi)^{-1} \circ d_0 (\lambda \rightarrow g(\lambda, \Phi(\xi)))
\]

and hence

\[
F_\xi'(0) = d_0 F_\xi \left( \frac{d}{d\lambda} \bigg|_{0} \right) = (d_{\pi(\xi)} \varphi)^{-1}(a_1, \ldots, a_n) = \xi.
\]

Thus

\[
A_M(\xi) = \limsup_{|\lambda| \to 0} \frac{\exp u_M(F_\xi(\lambda), \pi(\xi))}{|\lambda|}, \quad \xi \in \pi^{-1}(U).
\]

Suppose \( A_M(\xi_0) < c \). By (12), there is a number \( r \in (0, r_0) \) such that

\[
\sup_{|\lambda|=r} u_M(F_\xi_0(\lambda), \pi(\xi_0)) - \log |\lambda| < \log c.
\]

This means that

\[
\sup_{|\lambda|=r} u_M(F_\xi_0(\lambda), \pi(\xi_0)) < \log rc.
\]

Because of upper semicontinuity of \( u_M \) and continuity of \( F \) and \( \pi \) one can find a neighbourhood \( V \) of \( \xi_0 \) such that \( V \subset V_0 \) and

\[
\sup_{|\lambda|=r} u_M(F_\xi(\lambda), \pi(\xi)) < \log rc, \quad \xi \in V.
\]

Therefore

\[
\sup_{|\lambda|=r} (u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|) < \log c.
\]

If \( \xi \) fixed in \( V \), the function \( (\lambda \rightarrow u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|)^* \) is subharmonic in the disc \( \{\lambda : |\lambda| < r_0 \} \). Thus by the maximum principle for subharmonic functions

\[
\limsup_{\lambda \to 0} (u_M(F_\xi(\lambda), \pi(\xi)) - \log |\lambda|) < \log c
\]

for every \( \xi \in V \). Consequently \( A_M < c \) in \( V \).

3. Proof of the theorems

The first statement of Theorem 1 follows directly from the definitions (9), (11) of the pseudo-metrics and from Lemma 2. Lemma 3 and Lemma 4 yield the second conclusion of the theorem.
The estimates (3), (4) in Theorem 2 follow from the definition of $u_\Omega$ and from Lemma 2.

Now assume that $u_\Omega \in \mathcal{F}(\Omega, p)$ and the equality holds in (4) for all $\xi \in T_p M$. Without loss of generality we may assume that $M$ is submanifold of $C^m$ for some $m$. Since $\Omega$ is relatively compact in $M$, it is contained in an open ball with centre at $p$ and a positive radius $r$. It is clear that $\log \|z - p\| - \log r \leq u_\Omega(z)$ for all $z \in \Omega$. Let $N$ be a connected one-dimensional complex submanifold of $\Omega$ such that $p \in N$ and $u|(N\setminus\{p\})$ is harmonic. Let $F : V \to M$ be a holomorphic parametrization of $N$ in a neighbourhood of $p$, such that $0 \in V \subset \mathbb{C}$ and $F(0) = p$. By applying (6) and the lower estimate for $u_\Omega$, we get

$$\langle \mathcal{L}(\exp 2u_\Omega)(p)F'(0), F'(0) \rangle \geq \lim_{\lambda \to 0} \frac{\|F(\lambda) - F(0)\|^2}{|r\lambda|^2} = \frac{\|F'(0)\|^2}{r^2} > 0.$$

Hence, (6) implies that

$$1 = \frac{\langle \mathcal{L}v(p)F'(0), F'(0) \rangle}{\langle \mathcal{L}(\exp 2u_\Omega)(p)F'(0), F'(0) \rangle} = \lim_{\lambda \to 0} \frac{(v \circ F)(\lambda)}{\exp(2u_\Omega \circ F)(\lambda)}.$$

Therefore the function $((u/\exp 2u_\Omega)|N)^*$ is subharmonic on $N$ and attains its maximal value at $p$. Thus the maximum principle implies that $v = \exp(2u_\Omega)$ on $N$. As $u_\Omega \in \mathcal{F}(\Omega, p)$, the same equality holds in $\Omega$.

If $n = 1$ and the equality holds in (3) for one point $z \in \Omega \setminus\{p\}$, then the subharmonic function $v/(\exp 2u_\Omega)$ has its maximum at $z$. Therefore—by the maximum principle—it is constant.

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