THREE-SPACE PROBLEMS FOR THE APPROXIMATION PROPERTIES

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Abstract. Let \( M \) be a closed subspace of a Banach space \( X \). We suppose that \( M \) has the B.A.P. and that \( M^\perp \) is complemented in \( X^* \). Then, if \( X/M \) has the B.A.P. (resp. the A.P.), the space \( X \) has the same property. There are similar results if \( M \) is a \( C_0 \) space. If \( X/M \) is an \( L_1 \) space, then \( X \) has the B.A.P. if and only if \( M \) has the B.A.P. We notice that the quotient algebra \( L(H)/K(H) \) (\( H \) infinite-dimensional Hilbert space) does not have the A.P.

1. Introduction

Let \( X \) be a Banach space, and \( M \) a closed subspace of \( X \). Assume that the spaces \( M \) and \( X/M \) have the bounded approximation property (B.A.P.); what can be said about \( X \)? It is known that this does not imply in general that \( X \) the approximation property (A.P.); Indeed W. B. Johnson and H. P. Rosenthal have shown in [6] that every separable space \( X \) contains a subspace \( M \) such that both \( M \) and \( X/M \) have a finite-dimensional decomposition. More recently, W. Lusky in [12] has shown that if \( X \) is separable and contains a subspace isomorphic to \( c_0 \), then there exists a subspace \( M \) of \( X \) with a basis such that \( X/M \) has a shrinking basis. However, some positive results can be obtained under simple additional assumptions.

A typical result is the following: If \( M \) is a closed subspace of a Banach space \( X \) such that \( M^\perp \) is complemented in \( X^* \), and if \( X/M \) has the B.A.P., then \( X \) has the B.A.P. if and only if \( M \) has the B.A.P. We also show that if \( M \) is an \( L_\infty \) space and \( X/M \) has the A.P. (resp. the B.A.P.), then \( X \) has the A.P. (resp. the B.A.P.). On the other hand, if \( X/M \) is an \( L_1 \) space, \( X \) has the B.A.P. if and only if \( M \) has the B.A.P. We deduce from Szankowski's result [13], that the quotient algebra \( L(H)/K(H) \) (\( H \) infinite-dimensional Hilbert space) does not have the A.P.

Notations. The space of bounded operators of a Banach space \( X \) is denoted by \( L(X) \), and the space of finite rank operators by \( R(X) \). For two Banach spaces \( X \) and \( Y \), the quotient space \( X/Y \) is denoted by \( X/Y \).

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X and Y, the tensor product $X \otimes Y$ endowed with the projective norm $\pi$ and completed will be denoted $X \otimes_\pi Y$. If $X^*$ is the dual of $X$, the $w^*$-topology on $L(X^*)$ is the topology of pointwise convergence on the canonical predual $X^* \otimes_\pi X$ of $L(X^*)$. The topology $w_{op}^*$ on $L(X^*)$ is the topology of pointwise convergence on the algebraic tensor product $X^* \otimes X$. Our reference on the approximation properties (originally defined in [4]) is [11, Section 1.e]. The $L_1$ and $L_\infty$ spaces are defined and studied in [10].

2. Results

Our first lemma is a classical perturbation argument (see [2]).

Lemma 2.1. Let $X$ be a Banach space. Then:

1) $X$ has the A.P. if and only if $\text{Id}_{X^*}$ belongs to the closure of $R(X^*)$ in $(L(X^*), w^*)$;

2) $X$ has the B.A.P. if and only if there exists $\lambda > 0$ such that $\text{Id}_{X^*}$ belongs to the closure of $\{R; R \in R(X^*), \|R\| \leq \lambda\}$ in $(L(X^*), w_{op}^*)$ (or in $(L(X^*), w^*)$).

Proof. (1) Assume that $X$ has the A.P. Let $(R_\alpha)$ be a net of finite rank operators from $X$ into $X$ such that $R_\alpha \to \text{Id}_X$ for the topology $\tau_k$ of compact convergence. By [11, Proposition 1.e.3] one has $(R_\alpha) \to (\phi(\text{Id}_X))$ for every $\phi \in X^* \otimes X$; hence $R_\alpha \to \text{Id}_X$.

Assume conversely that $\text{Id}_{X^*}$ belongs to the closure of $R(X^*)$ in $(L(X^*), w^*)$. Let $U \in R(X^*)$. One verifies easily that there exists a net $(T_\alpha)$ in $R(X)$ such that $T_\alpha \to U$. Thus, there exists a net $(V_\alpha)$ in $R(X)$ such that $V_\alpha \to \text{Id}_X$. Hence, $V_\alpha$ converges to $\text{Id}_X$ for the weak topology of $(L(X), \tau_k)$ and a convex combination argument shows that $X$ has the A.P.

(2) If $X$ has the B.A.P., it is clear that there exists $\lambda > 0$ such that $\text{Id}_{X^*}$ belongs to the closure of $\{R; R \in R(X^*), \|R\| \leq \lambda\}$ in $(L(X^*), w^*)$. The converse is exactly Theorem 1 of [2].

Observe finally that by compactness, the topologies $w^*$ and $w_{op}^*$ coincide on the bounded subsets of $L(X^*)$. □

The next results will show the main tools for obtaining positive results in the "three-space" situation.

Lemma 2.2. Let $X$ be a Banach space, and $M$ a closed subspace of $X$ such that $X/M$ has the A.P. If there exists a bounded net $(T_\alpha)$ in $R(X)$ such that

$$\langle T_\alpha(x), x^* \rangle \to \langle x, x^* \rangle \quad \text{for each } x \in M \text{ and each } x^* \in X^*,$$

then $X$ has the A.P.

Proof. The net $(T_\alpha^*)$ is a bounded net in the dual space $L(X^*)$. Let $U$ be a $w^*$-cluster point of $(T_\alpha^*)$. Clearly $\langle x, U(x^*) \rangle = \langle x, x^* \rangle$ for each $x \in M$ and each $x^* \in X^*$. Then, if $j$ is the canonical map from $M^\perp$ to $X^*$, there exists an operator $D$ from $X^*$ to $M^\perp$ such that $U - \text{Id}_{X^*} = jD$.  

By assumption, $X/M$ has the A.P., hence there is a net $(S_\beta)$ in $R(X/M)$ such that $(S_\beta^*)$ satisfies $S_\beta^* \rightharpoonup \text{Id}_{M^\perp}$ in $L(M^\perp)$. If we let $V_\beta = jS_\beta D$, we have $V_\beta^* \rightharpoonup jD$ in $L(X^*)$. This shows that $\text{Id}_{X^*} = U - jD$ belongs to the $w^*$-closure of the set $(T_\alpha^* - V_\beta)$; hence by 2.1(1), $X$ has the A.P. □

In the case where $X/M$ is assumed to have the B.A.P., we can state

**Lemma 2.3.** Let $X$ be a Banach space and $M$ a closed subspace of $X$ such that $X/M$ has the B.A.P. Then the following are equivalent:

1. $X$ has the B.A.P.;
2. There exists a bounded net $(T_\alpha)$ in $R(X)$ such that
   \[
   \forall x \in M, \forall x^* \in X^*, \langle T_\alpha(x), x^* \rangle \to \langle x, x^* \rangle.
   \]

**Proof.** (1) $\Rightarrow$ (2) is clear by restriction.

(2) $\Rightarrow$ (1). We repeat the proof of 2.2 with the same notation. Since $X/M$ has the B.A.P., the net $(S_\beta)$ may be taken bounded; then $(V_\beta)$ is bounded and $\text{Id}_{X^*}$ is in the $w^*$-closure of a bounded subset of $R(X^*)$. We conclude by 2.1(2). □

Let us now state the main result of this note.

**Theorem 2.4.** Let $X$ be a Banach space, and $M$ a closed subspace of $X$ such that $M^\perp$ is complemented in $X^*$. Then we have:

1. If $X$ has the A.P. (resp. the B.A.P.), $M$ has the A.P. (resp. the B.A.P.);
2. If $M$ has the B.A.P., then $X/M$ has the A.P. implies that $X$ has the A.P.,
   
   $X/M$ has the B.A.P. implies that $X$ has the B.A.P.

**Proof.** Let $i$ be the canonical map from $M$ to $X$. Since $M^\perp$ is complemented in $X^*$, there exists an operator $\sigma$ from $M^*$ to $X^*$ such that $i^*\sigma = \text{Id}_{M^*}$.

(1) Let $(T_\alpha)$ be a net in $R(X^*)$ such that $T_\alpha \rightharpoonup \text{Id}_{X^*}$ in $L(X^*)$. We consider the operators $W_\alpha = i^*T_\alpha \sigma$; it is clear that $W_\alpha \in R(M^*)$ and that $W_\alpha \rightharpoonup \text{Id}_{M^*}$ in $L(M^*)$. Moreover, $\|W_\alpha\| \leq \|T_\alpha\| \|\sigma\|$. Hence, the net $(W_\alpha)$ is bounded if $(T_\alpha)$ is bounded. Lemma 2.1 concludes the proof.

(2) Let $(R_\alpha)$ be a bounded net in $R(M)$ such that $R_\alpha m \to m$, for every $m \in M$. Each operator $(R_\alpha)$ can be written

\[
R_\alpha = \sum_{t=1}^{n(\alpha)} m_{t,\alpha}^* \otimes m_{t,\alpha}, \quad m_{t,\alpha}^* \in M^*, m_{t,\alpha} \in M.
\]

We define $S_\alpha \in R(X)$ by

\[
S_\alpha = \sum_{t=1}^{n(\alpha)} \sigma(m_{t,\alpha}^*) \otimes m_{t,\alpha}.
\]
It is clear that for every $m \in M$ and every $x^* \in X^*$
\[ \langle S_\alpha(m), x^* \rangle = \langle R_\alpha(m), x^* \rangle \rightarrow \langle m, x^* \rangle. \]
Moreover, $\|S_\alpha\| \leq \|\sigma\| \cdot \|R_\alpha\|$, and the net $(S_\alpha)$ is bounded. Now Lemmas 2.2 and 2.3 conclude the proof. \( \Box \)

We describe now a few consequences of this result. Our first observation deals with subspaces of $X$ containing $M$.

**Corollary 2.5.** Let $M$ and $Y$ be two subspaces of $X$ such that $M \subset Y \subset X$. Suppose $M^\perp$ is complemented in $X^*$. Then, the orthogonal of $M$ in $Y^*$ is complemented in $Y^*$. Hence if $M$ does not have the A.P. (resp. the B.A.P.), no space $Y$ between $M$ and $X$ has the A.P. (resp. the B.A.P.).

**Proof.** If we write $X^* = M^\perp \oplus Z$ then we have
\[ Y^* = X^*/Y^\perp = (M^\perp/Y^\perp) \oplus Z \]
and the space $M^\perp/Y^\perp$ is precisely the orthogonal of $M$ in $Y^*$. The conclusion follows by 2.4(1). \( \Box \)

**Example 2.6.** Let $X$ be a Banach space, and $G = X^U$ an ultrapower of $X$ (see, for instance [1 or 5]). If $x = (x_i)$ is an element of $G$, we can define a map $\sigma$ from $X^*$ to $G^*$ by $\langle x, \sigma(f) \rangle = \lim_{U}(\langle x_i, f \rangle)$ for each $f$ of $X^*$. It is clear that $\sigma$ is a right inverse of the canonical map from $G^*$ to $X^*$. Then $X^\perp$ is complemented in $G^*$. Hence, Theorem 2.4 applies to this situation. Let $F$ be a subspace of $G$ such that $X \subset F \subset G$; by Corollary 2.5 we obtain that if $X$ does not have the A.P. (resp. the B.A.P.), it is the same for $F$. For a similar connection between finite representability and extensions, see [8].

The above applies for instance to any Banach space $F$ such that $X \subset F \subset X^{**}$. In the case $F = X^{**}$, we can deduce from [7] more precise results, namely:

**Corollary 2.7.** Let $X$ be a Banach space. Let us call $(P)$ one of the properties:

(i) $Y$ has a basis;
(ii) $Y$ has an F.D.D.;
(iii) $Y$ is a $\pi$-space (see [7, p. 489]);
(iv) $Y$ has the B.A.P.

Then if $X$ and $X^{**}/X$ have $(P)$, $X^{**}$ and $X^*$ have $(P)$.

**Proof.** If $X$ and $X^{**}/X$ have the B.A.P., then $X^{**}$ has the B.A.P. by 2.4(2), and thus $X^*$ has the B.A.P. [11, Theorem 1.e.7] and (iv) is proved. Now the conclusion follows:

(a) if $(P)$ is (i), from [7, Theorem 1.4.(b)];
(b) if $(P)$ is (ii), from [7, Theorem 1.3]);
(c) if $(P)$ is (iii), from [7, Corollary 4.8]. \( \Box \)

The next observation is a consequence of an important result of Szankowski (see [13]).
Corollary 2.8. Let $H$ be an infinite-dimensional Hilbert space, and $K(H)$ be the space of compact operators on $H$. Then the quotient algebra $L(H)/K(H)$ does not have the A.P.

Proof. Since $L(H) = K(H)^{**}$, $K(H)^{\perp}$ is complemented in $L(H)^*$. The space $K(H)$ has the B.A.P. On the other hand, by [13], $L(H)$ does not have the A.P. Therefore, 2.4(2) concludes the proof. □

Let us finally show

Corollary 2.9. Let $M$ be a closed subspace of the Banach space $X$. Then:

1. If $M$ is an $\mathcal{L}_\infty$ space and $X/M$ has the A.P. (resp. the B.A.P.), then $X$ has the A.P. (resp. the B.A.P.);
2. If $X/M$ is an $\mathcal{L}_1$ space, then $X$ has the B.A.P. if and only if $M$ has the B.A.P.

Proof. (1) If $M$ is an $\mathcal{L}_\infty$ space (see [10]), then $M$ has the B.A.P. Moreover, there exists a constant $K$ such that every finite rank operator $R: M \to M$ admits an extension $\tilde{R}: X \to M$ of finite rank, with $\|\tilde{R}\| \leq K \cdot \|R\|$. Hence there exists a bounded net $(T_\alpha)$ in $R(X)$ such that $(T_\alpha(x))$ converges weakly to $x$ for every $x \in M$. Lemmas 2.2 and 2.3 conclude the proof.

(2) If $X/M$ is an $\mathcal{L}_1$ space, then $M^\perp$ is a dual $\mathcal{L}_\infty$ space and thus $M^\perp$ is complemented in $X^*$. Moreover, $X/M$ has the B.A.P. The result now follows by 2.4. □

Remarks. (1) Let $E$ be a separable Banach space. By [9], there exists a space $Y$ such that $Y^{**}$ has a basis and $Y^{**}/Y$ is isomorphic to $E$. If we choose $E$ to be a separable Banach space without the A.P. (see [3]), we have an example of a couple of spaces $Y = M$, $Y^{**}/Y$ is isomorphic to $E$ where $M$ is complemented in $X^*$, $X$ and $M$ have the B.A.P. but $X/M$ does not have the A.P. (2) There is apparently no known example of a Banach space $X$ with the A.P. containing a closed subspace $M$ without the A.P., but such that $X/M$ has the A.P. (3) If $M$ is an $M$-ideal in $X$, then obviously $M^\perp$ is complemented in $X^*$ and thus 2.4 applies. Let us mention that under that assumption if $X/M$ is separable and has the B.A.P., then $M$ is complemented in $X$ (see [14]).

References


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