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APPROXIMATIONS AND FIXED POINTS FOR CONDENSING NON-SELF-MAPS DEFINED ON A SPHERE

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ABSTRACT. In this paper, we investigate the validity of an interesting theorem of Ky Fan [Theorem 2, Math. Z. **112** (1969), 234–240] defined on a sphere (the boundary of a closed ball) in an infinite-dimensional Banach space. We will prove that it is true for a continuous condensing map with suitable conditions posed. As applications of our theorem, some fixed point theorems of continuous condensing non-self-maps defined on a sphere are derived. Our results generalize some results of R. Nussbaum [10] and P. Massatt [8].

Most fixed point theorems in Banach spaces deal with some classes of maps defined on a compact convex (or star-shaped) subset, a closed bound convex (or star-shaped) subset, or a closed bound subset with nonempty interior. What about the domain of a function which is neither of the above cases? The simplest interesting case would be a sphere (the boundary of a closed ball). It is clear that a continuous self-map defined on a sphere may not have fixed points; for example, a rotation of a sphere (or circle) in a plane. Nussbaum [10] proved that a continuous k-set-contractive map from a sphere into a sphere has a fixed point, if the dimension of the Banach space is infinite. Recently, Massatt [8] generalized this result to continuous condensing maps. For definitions of k-set-contractive and condensing maps, see for example [9 or7] . Generalizations of fixed point theory from self-maps to non-self-maps has been a very active topic in nonlinear functional analysis in the past two decades. Does a continuous condensing non-self-map defined on a sphere have a fixed point? Under which conditions?

On the other hand, Fan [3] proved the following interesting theorem:

Let K be a nonempty compact convex subset of a normed linear space X. Let f be a continuous map from K into X; then there exists a point u in K such that $\|u - f(u)\| = d(f(u), K)$.

The author [5] proved that it is true for a continuous condensing map defined on a closed ball of a Banach space. Is this still true for a continuous condensing map defined on a sphere of an infinite-dimensional Banach space? Under which conditions?

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In this paper, we will prove that the second question is true under appropriate conditions (Theorem 1 below). As applications of our theorem, we derive fixed point theorems for continuous condensing non-self-maps defined on a sphere under suitable conditions, which answer the first question and also generalize Massatt's result [8].

We also remark that for a continuous condensing (even more generally, 1-set-contractive) map defined on a closed bounded convex subset of a Hilbert space, the above result of Fan is still true; see the author and Yen [5–7].

Throughout this paper, we will denote

$$S_r = \{x \in X \mid \|x\| = r\}, \qquad B_r = \{x \in X \mid \|x\| < r\},$$
$$I_A(x) = \{x + c(z - x) \mid \text{ for some } z \in A, \text{ some } c > 0\},$$

$\overline{D}$ the closure of D, where X is a Banach space, r a positive number, A a convex set in X and D a set in X. It is clear that $A \subset I_A(x)$.

Now, we prove our main theorems.

Theorem 1. *Let S_r be a sphere with center at the origin and radius r in an infinite-dimensional Banach space X. Let f be a continuous condensing map from S into X. If*

$$\|f(x)\| \geq r \quad \textit{for each } x \in S_r,$$

then there exists a point $u \in S_r$ such that

$$\|u - f(u)\| = d(f(u), S_r) = d(f(u), \overline{B_r}).$$

Proof. Define

$$R(x) = \begin{cases} x, & \text{if } \|x\| \leq r, \\ rx/\|x\|, & \text{if } \|x\| \geq r. \end{cases}$$

From Nussbaum [9, Corollary 1], R is a continuous 1-set-contractive map from X onto $\overline{B_r}$. Let $F(x) = R \circ f(x)$, then F is also a continuous condensing map. Since $\|f(x)\| \geq r$ for each $x \in S_r$, we have

$$\|F(x)\| = \left\| \frac{rf(x)}{\|f(x)\|} \right\| = r,$$

this implies $F(x) \in S_r$ and $F: S_r \to S_r$. From Massatt [8], F has a fixed point in S_r, say u. Therefore

$$\|u - f(u)\| = \|F(u) - f(u)\| = \|R(f(u)) - f(u)\|$$
$$= \left\| \frac{rf(u)}{\|f(u)\|} - f(u) \right\| = \|f(u)\| - r.$$

For any $x \in S_r$ or $x \in \overline{B_r}$, we have

$$\|f(u)\| - r \leq \|f(u)\| - \|x\| \leq \|f(u) - x\|.$$

Hence

$$\|u - f(u)\| = d(f(u), S_r) = d(f(u), \overline{B_r}). \quad \square$$

Theorem 2. *Let S_r, B_r, f, and X be defined as in Theorem* 1. *Moreover, let f satisfy any one of the following conditions*:

(i) *For each $x \in S_r$, with $x \neq f(x)$, there exists y in $I_{\overline{B_r}}(x)$ such that*

$$\|y - f(x)\| < \|x - f(x)\|.$$

(ii) *f is weakly inward (i.e., $f(x) \in \overline{I_{\overline{B_r}}(x)}$ for each $x \in S_r$).*

(iii) $\|f(x) - x\|^2 \geq \|f(x)\|^2 - r^2$, *for each $x \in S_r$.*

Then f has a fixed point in S_r.

Proof. From Theorem 1, there exists a point $u \in S_r$ such that

$$\|u - f(u)\| = d(f(u), S_r) = d(f(u), \overline{B_r}). \tag{1}$$

If f satisfies (i), and $u \neq f(u)$, then there exists y in $I_{\overline{B_r}}(u)$ such that $\|y - f(u)\| < \|u - f(u)\|$. Since $y \in I_{\overline{B_r}}(u)$, there exists $z \in \overline{B_r}$, $c > 0$ such that $y = u + c(z - u)$. Actually, $c > 1$, otherwise $y \in \overline{B_r}$ which contradicts (1). Since

$$z = u + \frac{1}{c}(y - u) = \left(1 - \frac{1}{c}\right)u + \frac{1}{c}y = (1 - \beta)u + \beta y,$$

where $0 < \beta = 1/c < 1$, we have

$$\begin{aligned}\|z - f(u)\| &\leq (1 - \beta)\|u - f(u)\| + \beta\|y - f(u)\| \\ &< (1 - \beta)\|u - f(u)\| + \beta\|u - f(u)\| \\ &= \|u - f(u)\|,\end{aligned}$$

which contradicts (1). Therefore $u = f(u)$. It is clear that if f satisfies (ii), then f satisfies (i). If f satisfies (iii), we will show that u is a fixed point of f. From (iii),

$$\|f(u) - u\|^2 \geq \|f(u)\|^2 - r^2. \tag{2}$$

Since $\|f(u)\| \geq r$, we have $f(u) \neq 0$ and $rf(u)/\|f(u)\| \in S_r$. Therefore, from (1),

$$\|u - f(u)\| \leq \left\| \frac{rf(u)}{\|f(u)\|} - f(u) \right\| = \|f(u)\| - r,$$

and

$$\|u - f(u)\|^2 \leq (\|f(u)\| - r)^2. \tag{3}$$

Combining (2) and (3), we have $\|f(u)\| \leq r$ and hence $\|f(u)\| = r$. From (1), u is a fixed point of f. □

Remark. Condition (i) of Theorem 2 was first stated by Browder [2]. Condition (ii) was first considered by Halpern [4], and (iii) was considered by Altman [1]. To the best of my knowledge, Theorem 2 has not been derived through other approaches. In other words, even Theorem 2 is new to the mathematical literature. We also note that Massatt's result [8] only appeared recently, in 1983. □

Corollary (MASSATT [8]). *Let X, S_r be defined as in Theorem* 1, *and let f be a continuous condensing map from S_r into S_r. Then f has a fixed point.*

Proof. Since $f(x) \in S_r \subset \overline{B_r} \subset I_{\overline{B_r}}(x)$, from Theorem 2(ii), f has a fixed point. □

Finally, we give an example of a continuous condensing map f, with $\|f(x)\| < r$, such that the conclusion of Theorem 1 is still true, but f has no fixed point in S_r.

Example. Let S_r be a sphere in a Banach space X (finite-dimensional or infinite-dimensional). Let $f: S_r \to X$ and $f(x) = x/2$ for each $x \in S_r$. Then f is a contraction and is a continuous condensing map. Clearly f has no fixed point in S_r. But for every $u \in S_r$, we have

$$\|u - f(u)\| = d(f(u), S_r).$$

In fact, $\|u - f(u)\| = \|u\|/2 = r/2$ and

$$\|x - f(u)\| = \|x - u/2\| \geq |\,\|x\| - \|u/2\|\,| = |r - r/2| = r/2,$$

for $x \in S_r$. Therefore $\|u - f(u)\| = d(f(u), S_r)$. Certainly, in this example,

$$d(f(u), S_r) \neq d(f(u), \overline{B_r}). \quad \square$$

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