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ON THE STRONG UNBOUNDED COMMUTANT OF AN $\mathscr{O}^*$-ALGEBRA

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ABSTRACT. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathscr{H}$. An $\mathscr{O}^*$-algebra $\mathscr{A}$ on $\mathscr{D}$ is a $*$-algebra of linear operators defined on $\mathscr{D}$ and leaving $\mathscr{D}$ invariant which contains the identity map I of $\mathscr{D}$. The involution in $\mathscr{A}$ is the map $A \to A^+ := A^* \restriction \mathscr{D}$ (see [1]). It is possible to define different types of unbounded commutants of an $\mathscr{O}^*$-algebra $\mathscr{A}$. We follow the definitions given in [2]. Let $L_{\mathscr{A}}(\mathscr{D}, \mathscr{H})$ be the vector space of all continuous linear mappings of $\mathscr{D}[t_{\mathscr{A}}]$ into $\mathscr{H}$ with respect to the graph topology $t_{\mathscr{A}}$ on $\mathscr{D}$ introduced by the operators from $\mathscr{A}$. Then the strong unbounded commutant is defined as

$$\mathscr{A}_s^c := \{T \in L_{\mathscr{A}}(\mathscr{D}, \mathscr{H}) \colon T\mathscr{D} \subset \mathscr{D}, TAx = ATx \text{ for all } x \in \mathscr{D} \text{ and } A \in \mathscr{A}\}.$$

$\mathscr{A}_s^c$ is an algebra, but in general however it will not be $*$-invariant (see [2]). In this paper we show that even worse can happen. We give an example of such an $\mathscr{O}^*$-algebra $\mathscr{A}$ and an operator $T \in \mathscr{A}_s^c$ such that $\mathscr{D}(T^*) = \{0\}$.

In particular this shows that the strong unbounded commutant of an $\mathscr{O}^*$-algebra may contain operators which are not closable. Furthermore the constructed $\mathscr{O}^*$-algebra $\mathscr{A}$ gives another example for an $\mathscr{O}^*$-algebra whose so-called form commutant $\mathscr{A}_f^c$ contains a sesquilinear form which is not an operator (see [2]).

Let $\mathscr{H}$ be a separable Hilbert space with scalar product $\langle \cdot , \cdot \rangle$. Let $\{e_1, e_2, e_3, \dots\}$ be an orthonormal basis of $\mathscr{H}$ and $\mathscr{D}$ the algebraic linear span of these basisvectors. So $\mathscr{D}$ is a dense subspace of $\mathscr{H}$. We will construct an algebra $\mathscr{A}$ of linear operators on $\mathscr{D}$ leaving $\mathscr{D}$ invariant and such that $I \in \mathscr{A}$ and $\mathscr{D} \subset \mathscr{D}(A^*)$ and $A^*\mathscr{D} \subset \mathscr{D}$ for all $A \in \mathscr{A}$. We will also construct an operator $T \in \mathscr{A}_s^c$ such that $\mathscr{D}(T^*) = \{0\}$. For the condition $T \in L_{\mathscr{A}}(\mathscr{D}, \mathscr{H})$ we simply need that there is an $A \in \mathscr{A}$ such that $\|Tx\| \le \|Ax\|$ for all $x \in \mathscr{D}$.

We will actually proceed by first constructing T such that $\mathscr{D}(T^*) = \{0\}$ in a standard way and then a symmetric operator $A \colon \mathscr{D} \to \mathscr{D}$ such that $TA = AT$ on $\mathscr{D}$ and $\|Tx\| \le \|Ax\|$ for all $x \in \mathscr{D}$.

1. **Definition.** Define a linear operator $T \colon \mathscr{D} \to \mathscr{D}$ by $Te_k = e_n$ if $k = 2^{n-1}p$ where p is odd. Because each vector e_n is the image under T of infinitely

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many vectors e_k, it is straightforward to check that $\mathscr{D}(T^*) = \{0\}$. We can also prove the following lemma.

2. **Lemma.** *For every* $x = \sum_{j=1}^{\infty} x_j e_j$ *in* $\mathscr{D}$ *we have*

$$\|Tx\| \leq \left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right)^{1/2} \left(\sum_{j=1}^{\infty} j^2 |x_j|^2\right)^{1/2}.$$

Proof. Let $A_n = \{2^{n-1}p : p \in \mathbf{N} \text{ and } p \text{ odd}\}$ for all $n \in \mathbf{N}$. Then $Te_k = e_n$ if $k \in A_n$. So we obtain

$$\begin{aligned}
\|Tx\| &= \left\|T \sum_j x_j e_j\right\| = \left\|T \sum_n \sum_{j \in A_n} x_j e_j\right\| \\
&= \left\|\sum_n \left(\sum_{j \in A_n} x_j\right) e_n\right\| = \left(\sum_n \left|\sum_{j \in A_n} x_j\right|^2\right)^{1/2} \\
&= \left(\sum_n \left|\sum_{j \in A_n} \frac{1}{j}(j x_j)\right|^2\right)^{1/2} \\
&\leq \left(\sum_n \left(\sum_{j \in A_n} \frac{1}{j^2}\right) \left(\sum_{j \in A_n} j^2 |x_j|^2\right)\right)^{1/2} \\
&\leq \left(\sum_j \frac{1}{j^2}\right)^{1/2} \left(\sum_n \sum_{j \in A_n} j^2 |x_j|^2\right)^{1/2} \\
&= \left(\sum_j \frac{1}{j^2}\right)^{1/2} \left(\sum_j j^2 |x_j|^2\right)^{1/2}.
\end{aligned}$$

We will now come to the main proposition.

3. **Proposition.** *There exists a symmetric operator* $A : \mathscr{D} \to \mathscr{D}$ *so that* $AT = TA$ *on* $\mathscr{D}$ *and* $\|Tx\| \leq \|Ax\|$ *for all* $x \in \mathscr{D}$.

Proof. We choose a sequence $c_1 > c_2 > c_3 > \cdots$ of real numbers such that $c_n \geq 1$ for all n.

Suppose that we have defined $Ae_1, Ae_2, \ldots, Ae_n$ in $\mathscr{D}$ such that

(i) $\langle Ae_j, e_k\rangle = \langle e_j, Ae_k\rangle \in \mathbf{R}$ for $j, k = 1, \ldots, n$;

(ii) $ATe_j = TAe_j$ for $j = 1, \ldots, n$;

(iii) $c_n^2 \sum_{j=1}^n j^2 |x_j|^2 \leq \|Ax\|^2$ if $x = \sum_{j=1}^n x_j e_j$.

It is clear that this is possible for $n = 2$. One could simply define $Ae_1 = c_1 e_1$ and $Ae_2 = 2c_2 e_2$. Then (i) follows immediately, (ii) follows from the fact that

$Te_1 = e_1$ and $Te_2 = e_2$ and (iii) uses $c_1 > c_2$. We will show that it is then possible to define Ae_{n+1} in $\mathcal{D}$ such that (i), (ii) and (iii) hold up to $n+1$.

By induction we will get a symmetric operator $A\colon \mathcal{D} \to \mathcal{D}$ which commutes with T on $\mathcal{D}$ and such that

$$\|Ax\|^2 \geq \sum_{j=1}^{\infty} j^2 |x_j|^2 \quad \text{if } x = \sum_{j=1}^{\infty} x_j e_j .$$

And then because of the lemma a multiple of A will satisfy the conditions of the proposition. So we will proceed by defining Ae_{n+1}. Now $Ae_{n+1} = \sum_{j=1}^{\infty} p_j e_j$ and we have to determine the sequence $\{p_j\}_{j=1}^{\infty}$.

We want $Ae_{n+1} \in \mathcal{D}$ so that only finitely many terms can be non zero. Condition (i) will fix the numbers $p_1, p_2, \ldots, p_n$. We then may choose all the other $p_{n+1}, p_{n+2}, \cdots$ freely in $\mathbf{R}$, and we will already have that

$$\langle Ae_j, e_k\rangle = \langle e_j, Ae_k\rangle \in \mathbf{R} \quad \text{for all } j, k = 1, \ldots, n+1 .$$

We now consider the second condition. We must have

$$ATe_{n+1} = TAe_{n+1} = \sum_{j=1}^{\infty} p_j Te_j .$$

Since $n \geq 2$ we have $Te_{n+1} = e_k$ for some $k \leq n$. So the left hand side of this equation is given. But because for all k we have infinitely many j's such that $Te_j = e_k$ it is possible to find numbers $p_{n+1}, p_{n+2}, \ldots, p_m$, many of them may be 0, so that

$$ATe_{n+1} = \sum_{j=1}^{m} p_j Te_j .$$

So suppose that we have chosen $m \geq n+1$ and $p_1, p_2, \ldots, p_m$ so that now also $ATe_{n+1} = TAe_{n+1}$ when $Ae_{n+1} = \sum_{j=1}^{m} p_j e_j$. Then let $p > 0$ and k odd and $k > m$ and put

$$Ae_{n+1} = \sum_{j=1}^{m} p_j e_j + pe_k - pe_{k+2} .$$

This clearly will have no effect on condition (i) since $k > m \geq n+1$. Because $Te_k = Te_{k+2} = e_1$ the vector TAe_{n+1} will not change by adding this term so that also condition (ii) will still be satisfied. Therefore it remains to show that k and p can be chosen such that also (iii) is satisfied. So let $x = \sum_{j=1}^{n+1} x_j e_j$ and suppose that $\sum_{j=1}^{n+1} |x_j|^2 = 1$.

Let E be the projection onto the subspace spanned by the vectors $e_1, e_2, \ldots, e_m$. Then we have

$$\begin{aligned}\|Ax\|^2 &= \left\| A\sum_{j=1}^{n} x_j e_j + Ax_{n+1}e_{n+1} \right\|^2 \\ &= \left\| A\sum_{j=1}^{n} x_j e_j + x_{n+1}EAe_{n+1} + x_{n+1}p(e_k - e_{k+2}) \right\|^2 .\end{aligned}$$

Now suppose that not only $k > m$ but that k is also larger than all indices that appear in the expansions of Ae_j with $j = 1, \ldots, n$. Then we have

$$\|Ax\|^2 = \left\| A\sum_{j=1}^{n} x_j e_j + x_{n+1}EAe_{n+1} \right\|^2 + 2p^2|x_{n+1}|^2 .$$

Let

$$\varepsilon = \min\left\{ \frac{1}{2}, \frac{\sqrt{3}}{2}\frac{c_n - c_{n+1}}{(n+1)c_{n+1} + \|EAe_{n+1}\|} \right\}.$$

If $|x_{n+1}| \leq \varepsilon$ then in particular $|x_{n+1}| \leq \frac{1}{2}$ and since $\sum_{j=1}^{n+1} |x_j|^2 = 1$ we have $\sum_{j=1}^{n} |x_j|^2 \geq 3/4$. Also if $|x_{n+1}| \leq \varepsilon$ then

$$\begin{aligned}|x_{n+1}|((n+1)c_{n+1} + \|EAe_{n+1}\|) &\leq \frac{\sqrt{3}}{2}(c_n - c_{n+1}) \\ &\leq (c_n - c_{n+1})\left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} \\ &\leq (c_n - c_{n+1})\left(\sum_{j=1}^{n} j^2|x_j|^2\right)^{1/2} \\ &\leq \left\| A\sum_{j=1}^{n} x_j e_j \right\| - c_{n+1}\left(\sum_{j=1}^{n} j^2|x_j|^2\right)^{1/2} .\end{aligned}$$

So

$$\begin{aligned}c_{n+1}\left(\sum_{j=1}^{n+1} j^2|x_j|^2\right)^{1/2} &\leq c_{n+1}(n+1)|x_{n+1}| + c_{n+1}\left(\sum_{j=1}^{n} j^2|x_j|^2\right)^{1/2} \\ &\leq \left\| A\sum_{j=1}^{n} x_j e_j \right\| - |x_{n+1}|\|EAe_{n+1}\| \\ &\leq \left\| A\sum_{j=1}^{n} x_j e_j + x_{n+1}EAe_{n+1} \right\| \leq \|Ax\| .\end{aligned}$$

If we now choose p such that

$$p \geq \frac{c_{n+1}}{\varepsilon\sqrt{2}} \left(\sum_{j=1}^{n+1} j^2 \right)^{1/2}$$

and if $|x_{n+1}| \geq \varepsilon$ then we have immediately

$$\begin{aligned} c_{n+1} \left(\sum_{j=1}^{n+1} j^2 |x_j|^2 \right)^{1/2} &\leq c_{n+1} \left(\sum_{j=1}^{n+1} j^2 \right)^{1/2} \leq \varepsilon p \sqrt{2} \\ &\leq |x_{n+1}| p \sqrt{2} \leq \|Ax\| . \end{aligned}$$

So we see that if p is large enough we will have for all $x \in \mathscr{D}$ that

$$c_{n+1} \left(\sum_{j=1}^{n+1} j^2 |x_j|^2 \right)^{1/2} \leq \|Ax\|$$

when $x = \sum_{j=1}^{n+1} x_j e_j$. This completes the proof of the proposition. The main result of this note is now an easy consequence of the previous proposition.

4. **Theorem.** *There exist an $\mathscr{O}^*$-algebra $\mathscr{A}$ and an operator $T \in \mathscr{A}_s^c$ such that $\mathscr{D}(T^*) = \{0\}$.*

Proof. Simply take for $\mathscr{A}$ the polynomial algebra generated by the symmetric operator A constructed in the proposition.

In this way the operator T is a nonclosable operator in the strong unbounded commutant $\mathscr{A}_s^c$ of the $\mathscr{O}^*$-algebra $\mathscr{A}$. In [2] are defined also the weak unbounded commutant

$$\mathscr{A}_w^c := \{T \in L_{\mathscr{A}}(\mathscr{D}, \mathscr{H}) \colon \langle TAx, y\rangle = \langle Tx, A^+ y\rangle \text{ for all } x, y \in \mathscr{D} \text{ and } A \in \mathscr{A}\}$$

and the form commutant

$$\mathscr{A}_f^c := \{c \in B_{\mathscr{A}}(\mathscr{D}, \mathscr{D}) \colon c(Ax, y) = c(x, A^+ y) \text{ for all } x, y \in \mathscr{D} \text{ and } A \in \mathscr{A}\}$$

of an $\mathscr{O}^*$-algebra $\mathscr{A}$ on $\mathscr{D}$. There $B_{\mathscr{A}}(\mathscr{D}, \mathscr{D})$ denotes the vector space of all continuous sesquilinear forms (linear in the first and conjugate-linear in the second variable) on $\mathscr{D}[t_{\mathscr{A}}] \times \mathscr{D}[t_{\mathscr{A}}]$.

$\mathscr{A}_f^c$ is invariant under the involution $c \to c^+$, there c^+ is defined by $c^+(x, y) := \overline{c(y, x)}$, $x, y \in \mathscr{D}$. An operator T on $\mathscr{D}$ is identified with the associated sesquilinear form $c_T(x, y) := \langle Tx, y\rangle$, $x, y \in \mathscr{D}$. So we have $\mathscr{A}_w^c \subset \mathscr{A}_f^c$ (and of course $\mathscr{A}_s^c \subset \mathscr{A}_w^c$). In [2] is constructed a selfadjoint $\mathscr{O}^*$-algebra $\mathscr{A}$ and a sesquilinear form in $\mathscr{A}_f^c$ which is not an operator. Now we consider the $\mathscr{O}^*$-algebra $\mathscr{A}$ and the operator $T \in \mathscr{A}_s^c$ constructed in the present paper.

In a similar spirit we can show that the sesquilinear form $c_T^+ \in \mathscr{A}_f^c$ is not an operator, that means there is no operator $T_1 \in \mathscr{A}_w^c$ such that $c_T^+ = c_{T_1}$.

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