ON THE STRONG UNBOUNDED COMMUTANT
OF AN C*-ALGEBRA

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Abstract. Let \( D \) be a dense subspace of a Hilbert space \( \mathcal{H} \). An \( C^* \)-algebra \( A \) on \( D \) is a *-algebra of linear operators defined on \( D \) and leaving \( D \) invariant which contains the identity map \( I \) of \( D \). The involution in \( A \) is the map \( A \to A^* := A^* \upharpoonright D \) (see [1]). It is possible to define different types of unbounded commutants of an \( \mathcal{C}^* \)-algebra \( A \). We follow the definitions given in [2]. Let \( L_A(D, \mathcal{H}) \) be the vector space of all continuous linear mappings of \( D \) into \( \mathcal{H} \) with respect to the graph topology \( t_A \) on \( D \) introduced by the operators from \( A \). Then the strong unbounded commutant is defined as

\[
A_{sc} := \{ T \in L_A(D, \mathcal{H}) : T\mathcal{D} \subset \mathcal{D}, TAx = ATx \text{ for all } x \in \mathcal{D} \text{ and } A \in A \}.
\]

\( A_{sc} \) is an algebra, but in general however it will not be *-invariant (see [2]).

In this paper we show that even worse can happen. We give an example of such an \( \mathcal{C}^* \)-algebra \( A \) and an operator \( T \in A_{sc} \) such that \( \mathcal{D}(T^*) = \{0\} \).

In particular this shows that the strong unbounded commutant of an \( \mathcal{C}^* \)-algebra may contain operators which are not closable. Furthermore the constructed \( \mathcal{C}^* \)-algebra \( A \) gives another example for an \( \mathcal{C}^* \)-algebra whose so-called form commutant \( A_f \) contains a sesquilinear form which is not an operator (see [2]).

Let \( \mathcal{H} \) be a separable Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \). Let \( \{ e_1, e_2, e_3, \ldots \} \) be an orthonormal basis of \( \mathcal{H} \) and \( D \) the algebraic linear span of these basis vectors. So \( D \) is a dense subspace of \( \mathcal{H} \). We will construct an algebra \( A \) of linear operators on \( D \) leaving \( D \) invariant and such that \( I \in A \) and \( D \subset \mathcal{D}(A^*) \) and \( A^*D \subset D \) for all \( A \in A \). We will also construct an operator \( T \in A_{sc} \) such that \( \mathcal{D}(T^*) = \{0\} \). For the condition \( T \in L_A(D, \mathcal{H}) \) we simply need that there is an \( A \in A \) such that \( \|Tx\| \leq \|Ax\| \) for all \( x \in D \).

We will actually proceed by first constructing \( T \) such that \( \mathcal{D}(T^*) = \{0\} \) in a standard way and then a symmetric operator \( A : D \to D \) such that \( TA = AT \) on \( D \) and \( \|Tx\| \leq \|Ax\| \) for all \( x \in D \).

1. Definition. Define a linear operator \( T : D \to D \) by \( T e_k = e_n \) if \( k = 2^{n-1}p \) where \( p \) is odd. Because each vector \( e_n \) is the image under \( T \) of infinitely...
many vectors $e_k$, it is straightforward to check that $D(T^*) = \{0\}$. We can also prove the following lemma.

2. Lemma. For every $x = \sum_{j=1}^{\infty} x_j e_j$ in $D$ we have

$$\|Tx\| \leq \left( \sum_{j=1}^{\infty} \frac{1}{j^2} \right)^{1/2} \left( \sum_{j=1}^{\infty} j^2 |x_j|^2 \right)^{1/2}.$$ 

Proof. Let $A_n = \{2^{n-1} p : p \in \mathbb{N} \text{ and } p \text{ odd} \}$ for all $n \in \mathbb{N}$. Then $Te_k = e_n$ if $k \in A_n$. So we obtain

$$\|Tx\| = \left\| T \sum_j x_j e_j \right\| = \left\| T \sum_{j \in A_n} x_j e_j \right\| = \left\| \sum_{n} \left( \sum_{j \in A_n} x_j \right) e_n \right\| = \left( \sum_{n} \left( \sum_{j \in A_n} \frac{1}{j} (jx_j) \right)^2 \right)^{1/2} \leq \left( \sum_{n} \left( \sum_{j \in A_n} \frac{1}{j^2} \right) \left( \sum_{j \in A_n} j^2 |x_j|^2 \right) \right)^{1/2} \leq \left( \sum_{j} \frac{1}{j^2} \right)^{1/2} \left( \sum_{n} \sum_{j \in A_n} j^2 |x_j|^2 \right)^{1/2} = \left( \sum_{j} \frac{1}{j^2} \right)^{1/2} \left( \sum_{j} j^2 |x_j|^2 \right)^{1/2}.$$ 

We will now come to the main proposition.

3. Proposition. There exists a symmetric operator $A : D \to D$ so that $AT = TA$ on $D$ and $\|Tx\| \leq \|Ax\|$ for all $x \in D$.

Proof. We choose a sequence $c_1 > c_2 > c_3 > \cdots$ of real numbers such that $c_n \geq 1$ for all $n$.

Suppose that we have defined $Ae_1, Ae_2, \ldots, Ae_n$ in $D$ such that

(i) $\langle Ae_j, e_k \rangle = \langle e_j, Ae_k \rangle \in \mathbb{R}$ for $j, k = 1, \ldots, n$;

(ii) $Ae_j = Te_j$ for $j = 1, \ldots, n$;

(iii) $c_n^2 \sum_{j=1}^{n} j^2 |x_j|^2 \leq \|Ax\|^2$ if $x = \sum_{j=1}^{n} x_j e_j$.

It is clear that this is possible for $n = 2$. One could simply define $Ae_1 = c_1 e_1$ and $Ae_2 = 2c_2 e_2$. Then (i) follows immediately, (ii) follows from the fact that
$Te_1 = e_1$ and $Te_2 = e_2$ and (iii) uses $c_1 > c_2$. We will show that it is then possible to define $Ae_{n+1}$ in $\mathcal{D}$ such that (i), (ii) and (iii) hold up to $n + 1$.

By induction we will get a symmetric operator $A : \mathcal{D} \to \mathcal{D}$ which commutes with $T$ on $\mathcal{D}$ and such that

$$\|Ax\|^2 \geq \sum_{j=1}^{\infty} j^2 |x_j|^2 \quad \text{if} \quad x = \sum_{j=1}^{\infty} x_je_j.$$ 

And then because of the lemma a multiple of $A$ will satisfy the conditions of the proposition. So we will proceed by defining $Ae_{n+1}$. Now $Ae_{n+1} = \sum_{j=1}^{\infty} p_je_j$ and we have to determine the sequence $\{p_j\}_{j=1}^{\infty}$.

We want $Ae_{n+1} \in \mathcal{D}$ so that only finitely many terms can be non zero. Condition (i) will fix the numbers $p_1, p_2, \ldots, p_n$. We then may choose all the other $p_{n+1}, p_{n+2}, \ldots$ freely in $\mathbb{R}$, and we will already have that

$$\langle Ae_j, e_k \rangle = \langle e_j, Ae_k \rangle \in \mathbb{R} \quad \text{for all} \; j, k = 1, \ldots, n + 1.$$

We now consider the second condition. We must have

$$ATe_{n+1} = T Ae_{n+1} = \sum_{j=1}^{\infty} p_j Te_j.$$ 

Since $n \geq 2$ we have $Te_{n+1} = e_k$ for some $k \leq n$. So the left hand side of this equation is given. But because for all $k$ we have infinitely many $j$'s such that $Te_j = e_k$ it is possible to find numbers $p_{n+1}, p_{n+2}, \ldots, p_m$, many of them may be 0, so that

$$ATe_{n+1} = \sum_{j=1}^{m} p_j Te_j.$$ 

So suppose that we have chosen $m \geq n + 1$ and $p_1, p_2, \ldots, p_m$ so that now also $ATe_{n+1} = T Ae_{n+1}$ when $Ae_{n+1} = \sum_{j=1}^{m} p_j e_j$. Then let $p > 0$ and $k$ odd and $k > m$ and put

$$Ae_{n+1} = \sum_{j=1}^{m} p_j e_j + pe_k - pe_{k+2}.$$ 

This clearly will have no effect on condition (i) since $k > m \geq n + 1$. Because $Te_k = Te_{k+2} = e_1$ the vector $T Ae_{n+1}$ will not change by adding this term so that also condition (ii) will still be satisfied. Therefore it remains to show that $k$ and $p$ can be chosen such that also (iii) is satisfied. So let $x = \sum_{j=1}^{n+1} x_je_j$ and suppose that $\sum_{j=1}^{n+1} |x_j|^2 = 1$. 
Let $E$ be the projection onto the subspace spanned by the vectors $e_1, e_2, \ldots, e_m$. Then we have

$$\|Ax\|^2 = \left\| A \sum_{j=1}^{n} x_j e_j + Ax_{n+1} e_{n+1} \right\|^2 = \left\| A \sum_{j=1}^{n} x_j e_j + x_{n+1} E Ae_{n+1} + x_{n+1} p(e_k - e_{k+2}) \right\|^2.$$

Now suppose that not only $k > m$ but that $k$ is also larger than all indices that appear in the expansions of $Ae_j$ with $j = 1, \ldots, n$. Then we have

$$\|Ax\|^2 = \left\| A \sum_{j=1}^{n} x_j e_j + x_{n+1} E Ae_{n+1} \right\|^2 + 2p^2 |x_{n+1}|^2.$$

Let

$$\epsilon = \min \left\{ \frac{1}{2}, \frac{\sqrt{3}}{2} \frac{c_n - c_{n+1}}{(n+1)c_{n+1} + \|E Ae_{n+1}\|} \right\}.$$

If $|x_{n+1}| \leq \epsilon$ then in particular $|x_{n+1}| \leq \frac{1}{2}$ and since $\sum_{j=1}^{n+1} |x_j|^2 = 1$ we have $\sum_{j=1}^{n} |x_j|^2 \geq 3/4$. Also if $|x_{n+1}| \leq \epsilon$ then

$$|x_{n+1}|((n+1)c_{n+1} + \|E Ae_{n+1}\|) \leq \frac{\sqrt{3}}{2}(c_n - c_{n+1})$$

$$\leq (c_n - c_{n+1}) \left( \sum_{j=1}^{n} |x_j|^2 \right)^{1/2}$$

$$\leq (c_n - c_{n+1}) \left( \sum_{j=1}^{n} j^2 |x_j|^2 \right)^{1/2}$$

$$\leq \left\| A \sum_{j=1}^{n} x_j e_j - c_{n+1} \left( \sum_{j=1}^{n} j^2 |x_j|^2 \right)^{1/2} \right\|.$$

So

$$c_{n+1} \left( \sum_{j=1}^{n+1} j^2 |x_j|^2 \right)^{1/2} \leq c_{n+1} (n+1)|x_{n+1}| + c_{n+1} \left( \sum_{j=1}^{n} j^2 |x_j|^2 \right)^{1/2}$$

$$\leq \left\| A \sum_{j=1}^{n} x_j e_j - |x_{n+1}| \|E Ae_{n+1}\| \right\|$$

$$\leq \left\| A \sum_{j=1}^{n} x_j e_j + x_{n+1} E Ae_{n+1} \right\| \leq \|Ax\|.$$
If we now choose \( p \) such that
\[
p \geq \frac{c_{n+1}}{\varepsilon \sqrt{2}} \left( \sum_{j=1}^{n+1} j^2 \right)^{1/2}
\]
and if \( |x_{n+1}| \geq \varepsilon \) then we have immediately
\[
c_{n+1} \left( \sum_{j=1}^{n+1} j^2 |x_j|^2 \right)^{1/2} \leq \varepsilon p \sqrt{2} 
\]
\[
\leq |x_{n+1}| p \sqrt{2} \leq \|Ax\|.
\]
So we see that if \( p \) is large enough we will have for all \( x \in \mathcal{D} \) that
\[
c_{n+1} \left( \sum_{j=1}^{n+1} j^2 |x_j|^2 \right)^{1/2} \leq \|Ax\|
\]
when \( x = \sum_{j=1}^{n+1} x_j e_j \). This completes the proof of the proposition. The main result of this note is now an easy consequence of the previous proposition.

4. **Theorem.** There exist an \( \mathcal{O}^\ast \)-algebra \( \mathcal{A} \) and an operator \( T \in \mathcal{A}^c \) such that \( \mathcal{D}(T^\ast) = \{0\} \).

**Proof.** Simply take for \( \mathcal{A} \) the polynomial algebra generated by the symmetric operator \( A \) constructed in the proposition.

In this way the operator \( T \) is a nonclosable operator in the strong unbounded commutant \( \mathcal{A}_s^c \) of the \( \mathcal{O}^\ast \)-algebra \( \mathcal{A} \). In [2] are defined also the weak unbounded commutant
\[
\mathcal{A}_w^c := \{ T \in L_{\omega}(\mathcal{D}, \mathcal{H}) : (TAx , y) = (Tx , A^+ y) \text{ for all } x , y \in \mathcal{D} \text{ and } A \in \mathcal{A} \}
\]
and the form commutant
\[
\mathcal{A}_f^c := \{ c \in B_{\omega}(\mathcal{D}, \mathcal{D}) : c(Ax , y) = c(x , A^+ y) \text{ for all } x , y \in \mathcal{D} \text{ and } A \in \mathcal{A} \}
\]
of an \( \mathcal{O}^\ast \)-algebra \( \mathcal{A} \) on \( \mathcal{D} \). There \( B_{\omega}(\mathcal{D}, \mathcal{D}) \) denotes the vector space of all continuous sesquilinear forms (linear in the first and conjugate-linear in the second variable) on \( \mathcal{D}[t_{\mathcal{A}^c}] \times \mathcal{D}[t_{\mathcal{A}^c}] \).

\( \mathcal{A}_f^c \) is invariant under the involution \( c \to c^+ \), there \( c^+ \) is defined by
\[
c^+(x , y) := c(y , x), \quad x , y \in \mathcal{D}.
\]
An operator \( T \) on \( \mathcal{D} \) is identified with the associated sesquilinear form \( c_T(x , y) := (Tx , y) \), \( x , y \in \mathcal{D} \). So we have \( \mathcal{A}_w^c \subset \mathcal{A}_f^c \) (and of course \( \mathcal{A}_s^c \subset \mathcal{A}_w^c \)). In [2] is constructed a selfadjoint \( \mathcal{O}^\ast \)-algebra \( \mathcal{A} \) and a sesquilinear form in \( \mathcal{A}_f^c \) which is not an operator. Now we consider the \( \mathcal{O}^\ast \)-algebra \( \mathcal{A} \) and the operator \( T \in \mathcal{A}_s^c \) constructed in the present paper.
In a similar spirit we can show that the sesquilinear form $c^+_T \in \mathcal{A}_F^c$ is not an operator, that means there is no operator $T_1 \in \mathcal{A}_W^c$ such that $c^+_T = c_{T_1}$.

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