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QUASI-SASAKIAN HOMOGENEOUS STRUCTURES ON THE GENERALIZED HEISENBERG GROUP $H(p, 1)$

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ABSTRACT. All the homogeneous structures on the generalized Heisenberg group $H(p, 1)$ are found, obtaining a one-parameter family of quasi-Sasakian homogeneous structures on this group.

In [AS], Ambrose and Singer give a characterization of the homogeneous Riemannian manifolds through a tensor field T of type $(1, 2)$ satisfying certain conditions (see §1). Afterwards, F. Tricerri and L. Vanhecke [TV] obtain a classification for the homogeneous Riemannian spaces into eight different classes by properties of the T's. Moreover, they determine all the homogeneous structures on the 3-dimensional Heisenberg group and prove that such structures belong to the class $T_2 \oplus T_3$. On the other hand, in [ChG] we have studied and characterized the almost contact metric homogeneous manifolds (i.e. almost contact metric manifolds with transitive almost contact isometry groups).

In this paper we find all the homogeneous structures on the $(2p + 1)$-dimensional generalized Heisenberg group $H(p, 1)$, endowed with its natural left-invariant metric. This group is an example of a connected, simply connected, two-step nilpotent, real Lie group of type H with one-dimensional center, and so it is a Heisenberg group [K]. Also, we obtain the transitive and effective groups of isometries on $H(p, 1)$ associated with a family of such homogeneous structures, and we give a one-parameter family of almost contact homogeneous structures $(T_\lambda; \varphi, \xi, \eta)$.

In §1, we give some results on almost contact metric manifolds and homogeneous structures on Riemannian manifolds. Beginning with §2 we determine all the homogeneous structures on $H(p, 1)$. Next, we characterize the homogeneous structures on this group of type $T_2 \oplus T_3$, obtaining also a large class of such structures. Moreover, the groups of isometries on $H(p, 1)$ associated with those examples are found. Finally, in §3, we give a new characterization

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for the quasi-Sasakian manifolds, i.e. normal almost contact metric manifolds with closed fundamental 2-form, in terms of its Riemannian connection, and we obtain a one-parameter family of almost contact homogeneous structures $(T_\lambda; \varphi, \xi, \eta)$ where T_λ are of type $T_2 \oplus T_3$ and (φ, ξ, η) is quasi-Sasakian.

1. Preliminaries

A $(2n+1)$-dimensional real differentiable manifold M of class C^∞ is said to have a (φ, ξ, η)-structure or an almost contact structure if it admits a field φ of endomorphisms of the tangent spaces, a vector field ξ, and a 1-form η satisfying

$$\eta(\xi) = 1, \qquad \varphi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity transformation, [B2].

Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. Such a (paracompact) manifold M with a (φ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where $X, Y \in \mathfrak{X}(M)$. Then M is said to have a (φ, ξ, η, g)-structure or an almost contact metric structure and g is called a compatible metric. The 2-form Φ on M defined by $\Phi(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of the almost contact metric structure. If ∇ is the Riemannian connection of g, then

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi), \qquad (\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z).$$

An almost contact metric structure (φ, ξ, η, g) is said to be normal if

$$(\nabla_X \varphi)Y - (\nabla_{\varphi X}\varphi)\varphi Y + \eta(Y)\nabla_{\varphi X}\xi = 0, \tag{1.1}$$

quasi-Sasakian if $d\Phi = 0$ and (φ, ξ, η) is normal, almost α-Sasakian if $\Phi = d\eta/\alpha$, $\alpha \in \mathbf{R} - \{0\}$, α-Sasakian if it is almost-α-Sasakian and normal.

For an extensive study of these structures we refer to [B1, B2, JV].

A connected Riemanniann manifold (M, g) is said to be homogeneous if there exists a connected Lie group G which acts on (M, g) as a transitive and effective group of isometries.

Ambrose and Singer [AS] proved that a connected, complete and simply connected Riemannian manifold (M, g) is homogeneous if and only if there exists a tensor field T of type $(1, 2)$ such that

$$\text{(AS)} \qquad \begin{cases} \text{(i)} & g(T_X Y, Z) + g(Y, T_X Z) = 0, \\ \text{(ii)} & (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X YZ} - R_{YT_X Z}, \\ \text{(iii)} & (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y} \end{cases}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Here ∇ denotes the Levi-Civita connection and R is the Riemannian curvature tensor of M. These conditions are equivalent to

(i) $\tilde{\nabla} g = 0$,
(ii) $\tilde{\nabla} R = 0$,
(iii) $\tilde{\nabla} T = 0$,

where $\widetilde{\nabla}$ is the connection determined by $\widetilde{\nabla} = \nabla - T$.

A homogeneous (Riemannian) structure on (M, g) is a tensor field T of type $(1, 2)$ which is a solution of the system (AS).

F. Tricerri and L. Vanhecke obtained in [TV] a classification of the homogeneous structures in eight different classes. These are:

(1)	symmetric	if $T = 0$,
(2)	T_1	if $T_{XYZ} = g(X, Y)\psi(Z) - g(X, Z)\psi(Y)$, $\psi \in \Lambda^1(M)$,
(3)	T_2	if $\mathfrak{S} T_{XYZ} = 0$ and $c_{12}(T) = 0$,
(4)	T_3	if $T_{XYZ} + T_{YXZ} = 0$,
(5)	$T_1 \oplus T_2$	if $\mathfrak{S} T_{XYZ} = 0$,
(6)	$T_1 \oplus T_3$	if $T_{XYZ} + T_{YXZ} = 2g(X, Y)\psi(Z) - g(X, Z)\psi(Y)$ $- g(Y, Z)\psi(X)$ with $\psi \in \Lambda^1(M)$,
(7)	$T_2 \oplus T_3$	if $c_{12}(T) = 0$,
(8)	$T_1 \oplus T_2 \oplus T_3$	no conditions,

where $\mathfrak{S}$ denotes the cyclic sum over $X, Y, Z \in \mathfrak{X}(M)$.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be almost contact homogeneous if (M, g) is homogeneous and φ is invariant under the action of the group. In [ChG] we have proved

Theorem 1.1. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact homogeneous manifold. Then, there exists a tensor field T of type $(1, 2)$ satisfying the conditions* (AS), *and furthermore*

$$\text{(iv)} \qquad \nabla_X \varphi = T_X \varphi - \varphi T_X, \quad \textit{for all } X \in \mathfrak{X}(M).$$

Conversely, if a connected, simply connected, complete almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ admits a tensor field T of type $(1, 2)$ satisfying (i)–(iv), *then $(M, \varphi, \xi, \eta, g)$ is an almost contact homogeneous manifold.*

From this theorem it follows that, for the almost contact homogeneous manifolds, ξ, η and Φ are invariant under the action of the group.

We shall call (T, φ, ξ, η) an almost contact homogeneous structure on $(M, \varphi, \xi, \eta, g)$ if it satisfies (i)–(iv) in Theorem 1.1.

2. Homogeneous structures on $H(p, 1)$

Let $H(p, 1)$ be the group of matrices of real numbers of the form

$$a = \begin{bmatrix} 1 & A & c \\ 0 & I_p & {}^tB \\ 0 & 0 & 1 \end{bmatrix}$$

where I_p denotes the identity $p \times p$ matrix, $A = (a_1, \dots, a_p)$, $B = (b_1, \dots, b_p) \in \mathbf{R}^p$, and $c \in \mathbf{R}$. $H(p, 1)$ is connected, simply connected nilpotent Lie

group of dimension $2p+1$, which is called a generalized Heisenberg group (see [H]). Moreover, $H(p,1)$ is a Heisenberg group [K].

A global system of coordinates (x_i, x_{p+i}, z), $1 \le i \le p$, on $H(p,1)$ is defined by

$$x_i(a) = a_i, \quad x_{p+i} = b_i, \quad z(a) = c \qquad (1 \le i \le p).$$

A basis for the left invariant 1-forms on $H(p,1)$ is given by

$$\alpha_i = dx_i, \qquad \alpha_{p+i} = dx_{p+i}, \qquad \gamma = dz - \sum_{j=1}^{p} x_j dx_{p+j}$$

and its dual basis of left invariant vector fields on $H(p,1)$ is given by

$$X_i = \frac{\partial}{\partial x_i}, \quad X_{p+i} = \frac{\partial}{\partial x_{p+i}} + x_i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z} \qquad (1 \le i \le p).$$

Define a left invariant metric on $H(p,1)$ by

$$g = \sum_{s=1}^{2p} \alpha_s \otimes \alpha_s + \gamma \otimes \gamma.$$

Then $\{X_s, Z\}$, $s = 1, \dots, 2p$, is an orthonormal frame with respect to g. Moreover, $[X_i, X_{p+i}] = Z$, $1 \le i \le p$, with the other bracket products equal to zero, and therefore it is easy to verify that, for the Riemannian connection ∇ of g,

$$\begin{aligned} \nabla_Z X_i &= \nabla_{X_i} Z = -\tfrac{1}{2} X_{p+i}, \\ \nabla_{X_i} X_{p+i} &= -\nabla_{X_{p+i}} X_i = \tfrac{1}{2} Z, \\ \nabla_Z X_{p+i} &= \nabla_{X_{p+i}} Z = \tfrac{1}{2} X_i, \end{aligned}$$

the other covariant derivatives being zero.

The connection forms are given by

$$\omega_{X_i Z}(X) = g(\nabla_X X_i, Z) = -\tfrac{1}{2}\alpha_{p+i}(X), \qquad \omega_{X_{p+i} Z} = \tfrac{1}{2}\alpha_i, \qquad \omega_{X_i X_{p+i}} = -\tfrac{1}{2}\gamma,$$

for any $X \in \mathfrak{X}(H(p,1))$, the remainder forms being zero. The curvature tensor is given by

$$\begin{aligned} &R(X_i, X_j, X_{p+i}, X_{p+j}) = \tfrac{1}{4}, \qquad i \ne j, \\ &R(X_i, X_{p+i}, X_i, X_{p+i}) = \tfrac{3}{4}, \\ &R(X_i, X_{p+i}, X_j, X_{p+j}) = \tfrac{1}{2}, \qquad i \ne j, \\ &R(X_i, X_{p+j}, X_j, X_{p+i}) = \tfrac{1}{4}, \qquad i \ne j, \\ &R(X_i, Z, X_i, Z) = R(X_{p+i}, Z, X_{p+i}, Z) = -\tfrac{1}{4}, \end{aligned} \tag{2.1}$$

and they are the only ones different to zero.

Next, we determine the homogeneous structures on $(H(p,1),g)$. Let T be a $(0,3)$-tensor field such that $T_{XYZ}+T_{XZY}=0$ for all $X,Y,Z\in\mathfrak{X}(H(p,1))$. By the condition (ii) of (AS), that is

$$\begin{aligned}(\nabla_X R)(Y,Z,W,V) = &-R(T_XY,Z,W,V)-R(Y,T_XZ,W,V)\\ &-R(Y,Z,T_XW,V)-R(Y,Z,W,T_XV),\end{aligned}$$

and replacing (Y,Z,W,V) by $(X_i,Z,X_j,X_{p+j}),(X_i,Z,X_{p+i},X_{p+j})$, $(X_i,X_{p+i},X_i,X_{p+j})$ and $(X_i,X_{p+i},X_{p+i},X_{p+j})$ we obtain, respectively,

$$\begin{aligned}&T_{XZX_{p+i}}=-\tfrac{1}{2}\alpha_i(X); && T_{XZX_i}=\tfrac{1}{2}\alpha_{p+i}(X);\\ &T_{XX_iX_j}=T_{XX_{p+i}X_{p+j}}; && T_{XX_iX_{p+j}}=T_{XX_jX_{p+i}};\end{aligned}\tag{2.2}$$

for all $X\in\mathfrak{X}(H(p,1))$. It is not hard to check that the condition (ii) of (AS) is equivalent to (2.2).

Put,

$$a_{ij}(X)=T_{XX_iX_j}=T_{XX_{p+i}X_{p+j}},\qquad b_{ij}(X)=T_{XX_iX_{p+j}}=T_{XX_jX_{p+i}}.$$

It follows that

$$a_{ij}=-a_{ji},\qquad b_{ij}=b_{ji}.$$

Let $\tilde\nabla$ be the connection determined by $\tilde\nabla=\nabla-T$. Then the connection forms of $\tilde\nabla$ are given by

$$\begin{aligned}&\tilde\omega_{X_iX_j}=\tilde\omega_{X_{p+i}X_{p+j}}=-a_{ij},\\ &\tilde\omega_{X_iX_{p+j}}=-b_{ij},\qquad i\neq j,\\ &\tilde\omega_{X_iX_{p+i}}=-(\tfrac{1}{2}\gamma+b_{ii}),\end{aligned}\tag{2.3}$$

the remainder forms being zero.

By the condition (iii) of (AS)

$$(\tilde\nabla_X T)(Y,Z,W)=0$$

and replacing Z,W by X_i,X_j, and X_i,X_{p+j}, we obtain, respectively

$$\begin{aligned}(\tilde\nabla_X a_{ij})Y+\sum_{k=1}^{p}(&a_{ik}(X)a_{kj}(Y)+a_{jk}(X)a_{ik}(Y)\\ &+b_{jk}(X)b_{ik}(Y)-b_{ik}(X)b_{kj}(Y))=0.\end{aligned}\tag{2.4}$$

$$\begin{aligned}(\tilde\nabla_X b_{ij})Y+\sum_{k=1}^{p}(&a_{ik}(X)b_{kj}(Y)-a_{ik}(Y)b_{kj}(X)\\ &+a_{jk}(X)b_{ik}(Y)-a_{jk}(Y)b_{ik}(X))=0.\end{aligned}\tag{2.5}$$

Furthermore, it follows that (2.4) and (2.5) are equivalent to the condition (iii) of (AS).

We conclude

Theorem 2.1. *All the homogeneous structures T on $(H(p,1),g)$ are given by*

$$
\begin{aligned}
2T = &\sum_{i=1}^{p}(\alpha_i \otimes \alpha_{p+i} \wedge \gamma + \alpha_{p+i} \otimes \gamma \wedge \alpha_i) \\
&+ \sum_{i,j=1}^{p}\{a_{ij} \otimes (\alpha_i \wedge \alpha_j + \alpha_{p+i} \wedge \alpha_{p+j}) + 2b_{ij} \otimes \alpha_i \wedge \alpha_{p+j}\},
\end{aligned} \tag{2.6}
$$

where the 1-forms a_{ij} and b_{ij} satisfy (2.4), (2.5) and $a_{ij} = -a_{ji}$, $b_{ij} = b_{ji}$.

Next, we classify the homogeneous structures on $(H(p,1),g)$. From [TV, Theorem 5.1], the connected Riemannian manifolds which admit a homogeneous structure $T \neq 0$ of type T_1 are of constant negative curvature. Thus, by (2.1), $(H(p,1),g)$ does not admit any homogeneous structure of type T_1.

Proposition 2.1. *A homogeneous structure T on $(H(p,1),g)$ is of type $T_2 \oplus T_3$ if and only if*

$$
\sum_{i=1}^{p}(a_{ij}(X_i) - b_{ij}(X_{p+i})) = \sum_{i=1}^{p}(a_{ij}(X_{p+i}) + b_{ij}(X_i)) = 0, \tag{2.7}
$$

for all j, $1 \leq j \leq p$.

Proof. By definition

$$
c_{12}(T)(X) = \sum_m T_{E_m E_m X}, \qquad X \in \mathfrak{X}(H(p,1)),
$$

for an arbitrary orthonormal basis $\{E_m\}$ on $H(p,1)$. Thus, using the orthonormal basis $\{X_s, Z\}$, $s = 1, \dots, 2p$, we have

$$
\begin{aligned}
c_{12}(T)(X_i) &= \sum_{j=1}^{p}(a_{ij}(X_i) - b_{ij}(X_{p+i})), \\
c_{12}(T)(X_{p+i}) &= \sum_{j=1}^{p}(a_{ij}(X_{p+i}) + b_{ij}(X_i)), \\
c_{12}(T)(Z) &= 0.
\end{aligned}
$$

This proves the proposition.

It is possible to obtain examples of homogeneous structures on $(H(p,1),g)$ satisfying (2.7). More specifically, let $T(r,s,t_1,\dots,t_p)$ be as in Theorem 2.1 with

$$
\begin{gathered}
a_{ij} = r\gamma \quad (i<j), \qquad a_{ji} = -r\gamma \quad (i>j), \\
b_{ij} = s\gamma = b_{ji}, \qquad i \neq j, \\
b_{ii} = t_i\gamma, \qquad i = 1, \dots, p,
\end{gathered}
$$

where $r, s, t_i \in \mathbf{R}$, $i = 1, \dots, p$. Trivially, by Proposition 2.1, this family of homogeneous structures is of type $T_2 \oplus T_3$. Further $T \in T_2$ if and only if

$t_i = -1$, for all $i = 1, \dots, p$, and $r = s = 0$; and $T \in T_3$ if and only if $t_i = \frac{1}{2}$ for all $i = 1, \dots, p$ and $r = s = 0$.

The Lie algebra $\underline{G}$ of the transitive and effective group G of isometries of $(H(p,1), g)$ associated with the homogeneous structure $T(r, s, t_1, \dots, t_p)$ is isomorphic to the direct sum $M \oplus K$, where K is the holonomy algebra of $\widetilde{\nabla} = \nabla - T$ and $M = T_x H(p,1)$, $x \in H(p,1)$. In what follows we shall take for x the origin o of $H(p,1)$. From [KN, Theorem 8.1], K is generated by the operators $(\widetilde{R}_o)_{XY}$, where $X, Y \in M$ and $\widetilde{R}$ is the curvature tensor of $\widetilde{\nabla}$, being the brackets of $M \oplus K$ the followings

$$
\begin{aligned}
&[X, Y] = (T_o)_X Y - (T_o)_Y X - (\widetilde{R}_0)_{XY}, \qquad X, Y \in M, \\
&[A, X] = A(X), \qquad X \in M,\ A \in K, \\
&[A, B] = AB - BA, \qquad A, B \in K.
\end{aligned} \tag{2.8}
$$

Using (2.3) and the structure equations we obtain

$$
\begin{aligned}
\widetilde{\Omega}_{X_i X_j} = \widetilde{\Omega}_{X_{p+i} X_{p+j}} &= r \sum_{k=1}^{p} \alpha_k \wedge \alpha_{p+k} \qquad (i < j), \\
\widetilde{\Omega}_{X_i X_{p+j}} &= s \sum_{k=1}^{p} \alpha_k \wedge \alpha_{p+k} \qquad (i \neq j), \\
\widetilde{\Omega}_{X_i X_{p+i}} &= \left(\frac{1}{2} + t_i\right) \sum_{j=1}^{p} \alpha_j \wedge \alpha_{p+j}, \\
\widetilde{\Omega}_{X_i Z} = \widetilde{\Omega}_{X_{p+i} Z} &= 0.
\end{aligned}
$$

Hence, K is generated by $B = (\widetilde{R}_o)_{X_i X_{p+i}}$, which has the following expression with respect to the basis $\{X_s, Z\}$,

$$
B = (\widetilde{R}_0)_{X_i X_{p+i}} = \begin{bmatrix}
0 & r & \cdots & r & (\frac{1}{2} + t_1) & s & \cdots & s & 0 \\
-r & 0 & & \vdots & s & \ddots & & \vdots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & & s & \vdots \\
-r & \cdots & -r & 0 & s & \cdots & s & (\frac{1}{2} + t_p) & 0 \\
-(\frac{1}{2} + t_1) & -s & \cdots & -s & 0 & r & \cdots & r & 0 \\
-s & \ddots & & \vdots & -r & 0 & & \vdots & \vdots \\
\vdots & & \ddots & -s & \vdots & & \ddots & \vdots & \vdots \\
-s & \cdots & -s & -(\frac{1}{2} + t_p) & -r & \cdots & -r & 0 & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
$$

From (2.8), the bracket product on $M \oplus K$ is given by

$$[X_i, X_j] = [X_{p+i}, X_{p+j}] = [X_i, X_{p+j}] = [B, Z] = 0, \qquad i \neq j;$$

$$[X_i, X_{p+i}] = Z + B,$$

$$[Z, X_i] = r\left(\sum_{i<j} X_j - \sum_{i>j} X_j\right) + s\sum_{i\neq j} X_{p+j} + (t_i + \tfrac{1}{2})X_{p+i},$$

$$[Z, X_{p+i}] = r\left(\sum_{i<j} X_{p+j} - \sum_{i>j} X_{p+j}\right) - s\sum_{i\neq j} X_j - (t_i + \tfrac{1}{2})X_i,$$

$$[B, X_i] = -\left\{r\left(\sum_{i<j} X_j - \sum_{i>j} X_j\right) + s\sum_{i\neq j} X_{p+j} + (t_i + \tfrac{1}{2})X_{p+i}\right\},$$

$$[B, X_{p+i}] = -r\left(\sum_{i<j} X_{p+j} - \sum_{i>j} X_{p+j}\right) + s\sum_{i\neq j} X_j + (t_i + \tfrac{1}{2})X_i.$$

If $t_i = -\frac{1}{2}$ for all $i = 1, \dots, p$, and $r = s = 0$, then the dimension of K is zero and the Lie algebra of G is isomorphic to the algebra of $H(p, 1)$. In the other case, if we put $V = B + Z$, it follows

$$[X_i, X_{p+i}] = V, \qquad [V, X_{p+i}] = [V, X_i] = [V, B] = 0.$$

Hence $\underline{G}$ is a $2(p+1)$-dimensional Lie algebra. The subalgebra generated by the vector fields X_s, V, $s = 1, \dots, 2p$, denoted by $\langle X_s, V\rangle$, is isomorphic to the Lie algebra of $H(p, 1)$. From the expressions of the brackets of these vector fields, the Lie algebra $\underline{G}$ is a semidirect sum of $\langle X_s, V\rangle$ and $\langle B\rangle$. More precisely,

$$\underline{G} = \langle X_s, V\rangle \times_\sigma \langle B\rangle$$

where σ is the representation of $\langle B\rangle$ on $\langle X_s, V\rangle$ given by

$$\begin{aligned}
\sigma(B)X_i &= -\left\{r\left(\sum_{i<j} X_j - \sum_{i>j} X_j\right) + s\sum_{j\neq i} X_{p+j} + \left(\frac{1}{2} + t_i\right)X_{p+i}\right\},\\
\sigma(B)X_{p+i} &= -r\left(\sum_{i<j} X_{p+j} - \sum_{i>j} X_{p+j}\right) + s\sum_{j\neq i} X_j + \left(\frac{1}{2} + t_i\right)X_i,\\
\sigma(B)V &= 0.
\end{aligned} \tag{2.9}$$

Thus, in this case, G is a semidirect product of $H(p, 1)$ with a one-dimensional Lie group. To describe this group, we identify $H(p, 1)$ with $\mathbf{C}^p \times \mathbf{R}$ through the application $(x_j, x_{p+j}, z) \to (w_j, t)$ where $w_j = x_j + ix_{p+j}$ and $t = z - \frac{1}{2}\sum_{j=1}^p x_j x_{p+j}$ which is an isomorphism of Lie groups, considering on $\mathbf{C}^p \times \mathbf{R}$ the structure of Lie group given by

$$(w_j, t)(w_j', t') = \left(w_j + w_j', t + t' + \frac{1}{2}\sum_{j=1}^p \operatorname{Im}(\bar{w}_j w_j')\right).$$

Then, from (2.9) G is the semidirect product $H(p,1)\times_{\psi} SO(2)$ where each $e^{i\theta}\in SO(2)$ acts on $H(p,1)$ by the matrix

$$\begin{bmatrix} e^{-i\theta} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & & e^{-i\theta} & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{2.10}$$

So, we conclude

Theorem 2.2. *Let* $(H(p,1),g)$ *be with the homogeneous structure* $T(r,s,t_1,\dots,t_p)$. *For* $t_i=-\frac{1}{2}$ $(i=1,\dots,p)$, $r=s=0$ *the corresponding group of isometries is* $H(p,1)$ *itself and for all the other values of the parameters it is* $H(p,1)\times_{\psi} SO(2)$, *where* ψ *is given by* (2.10).

3. Quasi-Sasakian homogeneous structures on $H(p,1)$

First, we start with a characterization for the quasi-Sasakian manifolds in terms of its Riemannian connection.

Lemma 3.1. *An almost contact metric manifold* (M,φ,ξ,η,g) *is quasi-Sasakian if and only if*

$$(\nabla_X\Phi)(Y,Z)=\eta(Y)(\nabla_{\varphi X}\eta)Z+\eta(Z)(\nabla_Y\eta)\varphi X, \tag{3.1}$$

for all $X,Y,Z\in\mathfrak{X}(M)$.

Proof. Let M be a quasi-Sasakian manifold. Then, from (1.1) we have

$$(\nabla_X\Phi)(\xi,Y)=(\nabla_{\varphi X}\Phi)(\xi,\varphi Y) \tag{3.2}$$

and

$$(\nabla_\xi\Phi)(Y,Z)=0, \tag{3.3}$$

for all $X,Y,Z\in\mathfrak{X}(M)$. Furthermore, since Φ is closed,

$$(\nabla_{\varphi X}\Phi)(\varphi Y,\varphi Z)=0 \quad\text{and}\quad (\nabla_X\Phi)(Y,\xi)=(\nabla_Y\Phi)(X,\xi). \tag{3.4}$$

Now, (3.1) follows from (3.2), (3.3) and (3.4). The converse is immediate.

Next, we give a family of quasi-Sasakian structures on $H(p,1)$. For it, let (φ,ξ,η,g) be an almost contact metric structure on $H(p,1)$ and φ_η^m the components of φ, with respect to the basis $\{X_s,Z\}$, $s=1,\dots,2p$. We assume that $\varphi_n^m=$ *constant* and $Z=\xi$, then

$$(\nabla_{X_i}\varphi)X_r=\tfrac{1}{2}\varphi_r^{p+i}\xi,$$
$$(\nabla_{X_{p+i}}\varphi)X_r=-\tfrac{1}{2}\varphi_r^{i}\xi,$$

where $1\le i\le p$ and $1\le r\le 2p$.

Also, we obtain

$$(\nabla_\xi \varphi)X_i = \frac{1}{2}\sum_{j=1}^{p}\{(\varphi_i^{p+j} + \varphi_{p+i}^j)X_j + (\varphi_{p+i}^{p+j} - \varphi_i^j)X_{p+j}\},$$

$$(\nabla_\xi \varphi)X_{p+i} = \frac{1}{2}\sum_{j=1}^{p}\{(\varphi_{p+i}^{p+j} - \varphi_i^j)X_j - (\varphi_{p+i}^j + \varphi_i^{p+j})X_{p+j}\}.$$

Hence $(\nabla_\xi \varphi)X = 0$, for all $X \in \mathfrak{X}(H(p,1))$, if and only if

$$\begin{aligned} \varphi_j^{p+i} &= \varphi_i^{p+j} \\ \varphi_{p+j}^{p+i} &= \varphi_j^i \end{aligned} \qquad (1 \le i, j \le p). \tag{3.5}$$

If we suppose that (3.5) is satisfied, using condition (3.1), it is not hard to check that (φ, ξ, η, g) is a quasi-Sasakian structure. Moreover,

$$\delta\Phi(\xi) = \sum_{i=1}^{p} \varphi_i^{p+i}.$$

So, we have established the following.

Theorem 3.3. *On $(H(p,1), g)$, the almost contact metric structures (φ, ξ, η, g), with φ_n^m = constant, $\xi = Z$ and $\eta = \gamma$, satisfying (3.5) are quasi-Sasakian. Moreover, they are α-Sasakian if and only if*

$$\varphi_i^{p+i} = \lambda, \quad \textit{for all } i \in \{1, \dots, p\} \ (\lambda \in \mathbf{R} - \{0\})$$

and the remainder components of φ are zero.

Finally, we shall obtain that the previous structures (φ, ξ, η) are the only almost contact structures such that $(T_\lambda; \varphi, \xi, \eta)$, $\lambda \neq -\frac{1}{2}$, are almost contact homogeneous structures, where

$$2T_\lambda = \sum_{i=1}^{p}[\alpha_i \otimes \alpha_{p+i} \wedge \gamma + \alpha_{p+i} \otimes \gamma \wedge \alpha_i + 2\lambda\gamma \otimes \alpha_i \wedge \alpha_{p+i}], \qquad \lambda \in \mathbf{R}.$$

Theorem 3.4. *Let (φ, ξ, η) be an almost contact structure on $H(p,1)$, with g compatible metric. Then*

(a) *If $\lambda = -\frac{1}{2}$, $(T; \varphi, \xi, \eta)$ is an almost contact homogeneous structure if and only if the components φ_n^m and ξ_m are constant.*

(b) *If $\lambda \neq -\frac{1}{2}$, $(T; \varphi, \xi, \eta)$ is an almost contact homogeneous structure if and only if the components φ_n^m are constant, $\xi = \pm Z$ and*

$$\varphi_j^{p+i} = \varphi_i^{p+j}, \qquad \varphi_{p+j}^{p+i} = \varphi_j^i.$$

Proof. From (2.3), the connection $\widetilde{\nabla}$ reduces to

$$\widetilde{\nabla}_X X_i = -\gamma(X)(\tfrac{1}{2} + \lambda)X_{p+i}, \qquad \widetilde{\nabla}_X X_{p+i} = \gamma(X)(\tfrac{1}{2} + \lambda)X_i, \qquad \widetilde{\nabla}_X Z = 0.$$

Then, condition (iv) of Theorem 1.1, i.e. $\widetilde{\nabla}\varphi = 0$, is equivalent, in this case, to

$$
\begin{aligned}
X(\varphi^{p+i}_{p+j}) &= -X(\varphi^{i}_{j}) = \gamma(X)(\tfrac{1}{2}+\lambda)(\varphi^{i}_{p+j} + \varphi^{p+i}_{j}),\\
X(\varphi^{p+i}_{j}) &= X(\varphi^{i}_{p+j}) = \gamma(X)(\tfrac{1}{2}+\lambda)(\varphi^{i}_{j} + \varphi^{p+j}_{p+i}),\\
X(\varphi^{2p+1}_{i}) &= \gamma(X)(\tfrac{1}{2}+\lambda)\varphi^{p+i}_{2p+1},\\
X(\varphi^{2p+1}_{p+i}) &= \gamma(X)(\tfrac{1}{2}+\lambda)\varphi^{2p+1}_{i}.
\end{aligned}
\tag{3.6}
$$

Also, if $\widetilde{\nabla}\varphi = 0$, it follows that

$$
\begin{aligned}
X(\xi_i) &= -\gamma(X)(\tfrac{1}{2}+\lambda)\xi_{p+i},\\
X(\xi_{p+i}) &= \gamma(X)(\tfrac{1}{2}+\lambda)\xi_i,\\
X(\xi_{2p+1}) &= 0.
\end{aligned}
\tag{3.7}
$$

From (3.6) and (3.7), it is easy to prove (a). Using (3.7) we have

$$X_i(\xi_m) = X_{p+i}(\xi_m) = 0, \qquad 1 \le i \le p,\ 1 \le m \le 2p+1,$$

and hence

$$\xi_m = \xi_m(x_{p+1}, \dots, x_{2p}, z); \qquad \frac{\partial \xi_m}{\partial x_{p+i}} = -x_i \frac{\partial \xi_m}{\partial z}.$$

Now, it is not difficult to check that $\xi = \pm Z$.

From (3.6) we obtain

$$X_i(\varphi^{p+j}_{p+k}) = X_i(\varphi^{j}_{k}) = 0, \qquad 1 \le j, k \le p,$$

and this gives

$$\varphi^{p+j}_{p+k} = \varphi^{p+j}_{p+k}(x_{p+1}, \dots, x_{2p}, z), \qquad \varphi^{j}_{k} = \varphi^{j}_{k}(x_{p+1}, \dots, x_{2p}, z). \tag{3.8}$$

We also have that

$$X_{p+i}(\varphi^{p+j}_{p+k}) = X_{p+i}(\varphi^{j}_{k}) = 0,$$

and so

$$\frac{\partial \varphi^{p+j}_{p+k}}{\partial x_{p+i}} = -x_i \frac{\partial \varphi^{p+j}_{p+k}}{\partial z}, \qquad \frac{\partial \varphi^{j}_{k}}{\partial x_{p+i}} = -x_i \frac{\partial \varphi^{p+j}_{p+k}}{\partial z}. \tag{3.9}$$

Combining (3.8) and (3.9) we deduce that φ^{p+j}_{p+k} and φ^{j}_{k} are constant and $\varphi^{p+i}_{j} = \varphi^{p+j}_{i}$.

In the same way we can see that $\varphi^{i}_{j} = \varphi^{p+i}_{p+j}$, which proves (b).

From Theorems 3.3 and 3.4 we obtain

Corollary 3.1. *If $\lambda \neq \frac{1}{2}$ and $(T; \varphi, \xi, \eta)$ is an almost contact homogeneous structure on $(H(p,1), g)$, then (φ, ξ, η) is quasi-Sasakian.*

REFERENCES

[AS] W. Ambrose and I. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.

[B1] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geom. **1** (1967), 331–345.

[B2] ——, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., vol. 509, Springer-Verlag, Berlin and New York, 1976.

[ChG] D. Chinea and C. Gonzalez, *An example of an almost cosymplectic homogeneous manifold*, Lecture Notes in Math., vol. 1209, Springer-Verlag, Berlin and New York, 1986.

[H] Y. Haraguchi, *Sur une généralisation des structures de contact*, Théses, Univ. du Haute Alsace, Mulhouse, 1981.

[JV] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J. **4** (1981), 1–27.

[K] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. I, Interscience, New York, 1963.

[TV] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes 83, Cambridge Univ. Press, 1983.

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